Black Hole Spectroscopy Tools for Waveform Modeling

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arXiv: 2502.03155, 2408.05300

THE CENTER OF GRAVITY
UNIVERSITY OF COPENHAGEN

Outline

- Introduction to ringdown
- Mode fitting recipe and waveform frames
- Importance of multiple modes and BMS frame fixing
- QNM amplitude extraction packages
- Building surrogate model `NRSur3dq8_RD`
- Future work and conclusions

Binary Black Holes

- Inspiral – two BHs orbit around their center of mass (Post-Newtonian)
- Merger– strong gravitational radiation (Numerical Relativity)
- Ringdown – remnant black hole rings down like a bell (Perturbation Theory)

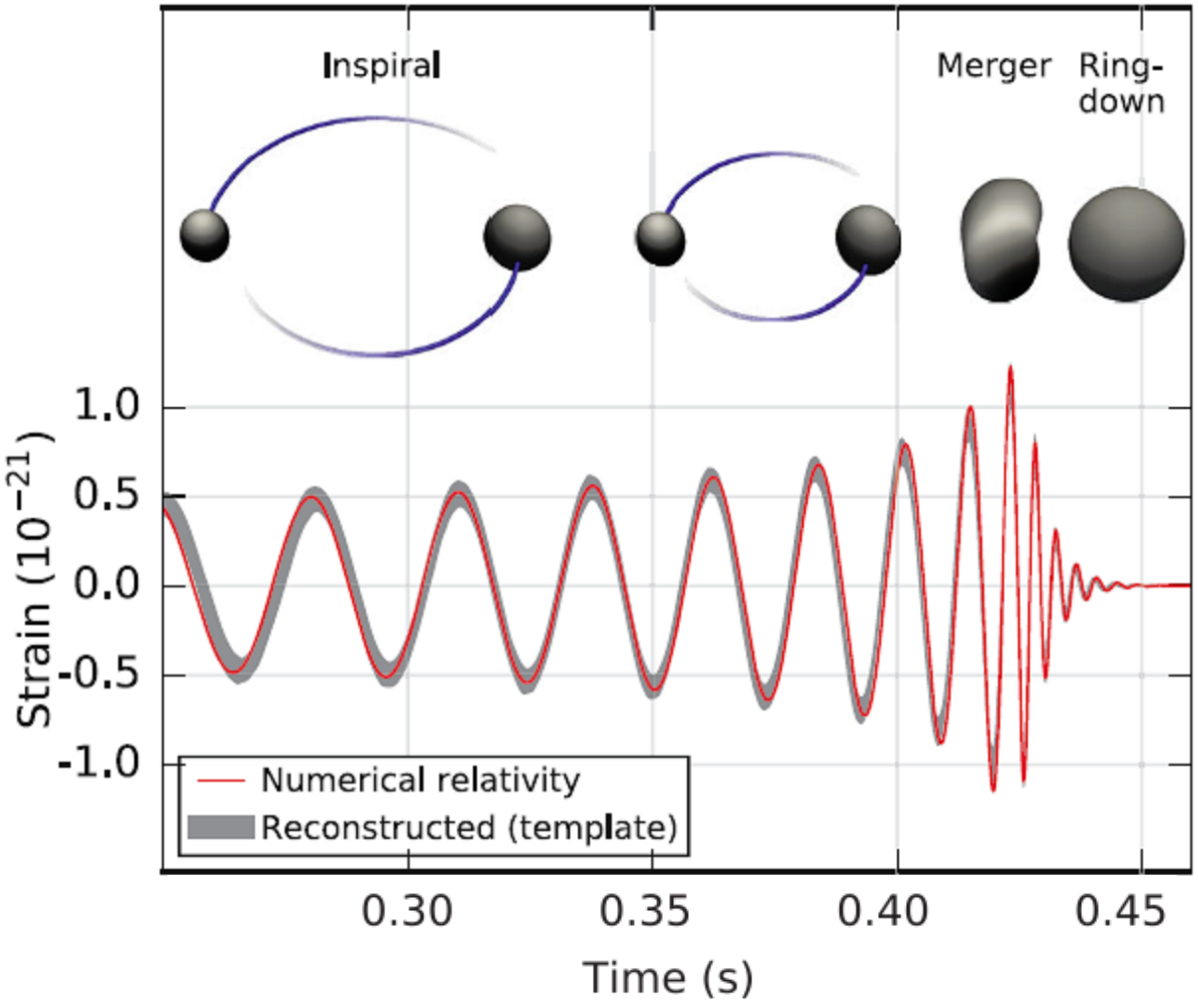

Image: Fig. 2 of LVC PRL **116**, 061102 (2016)

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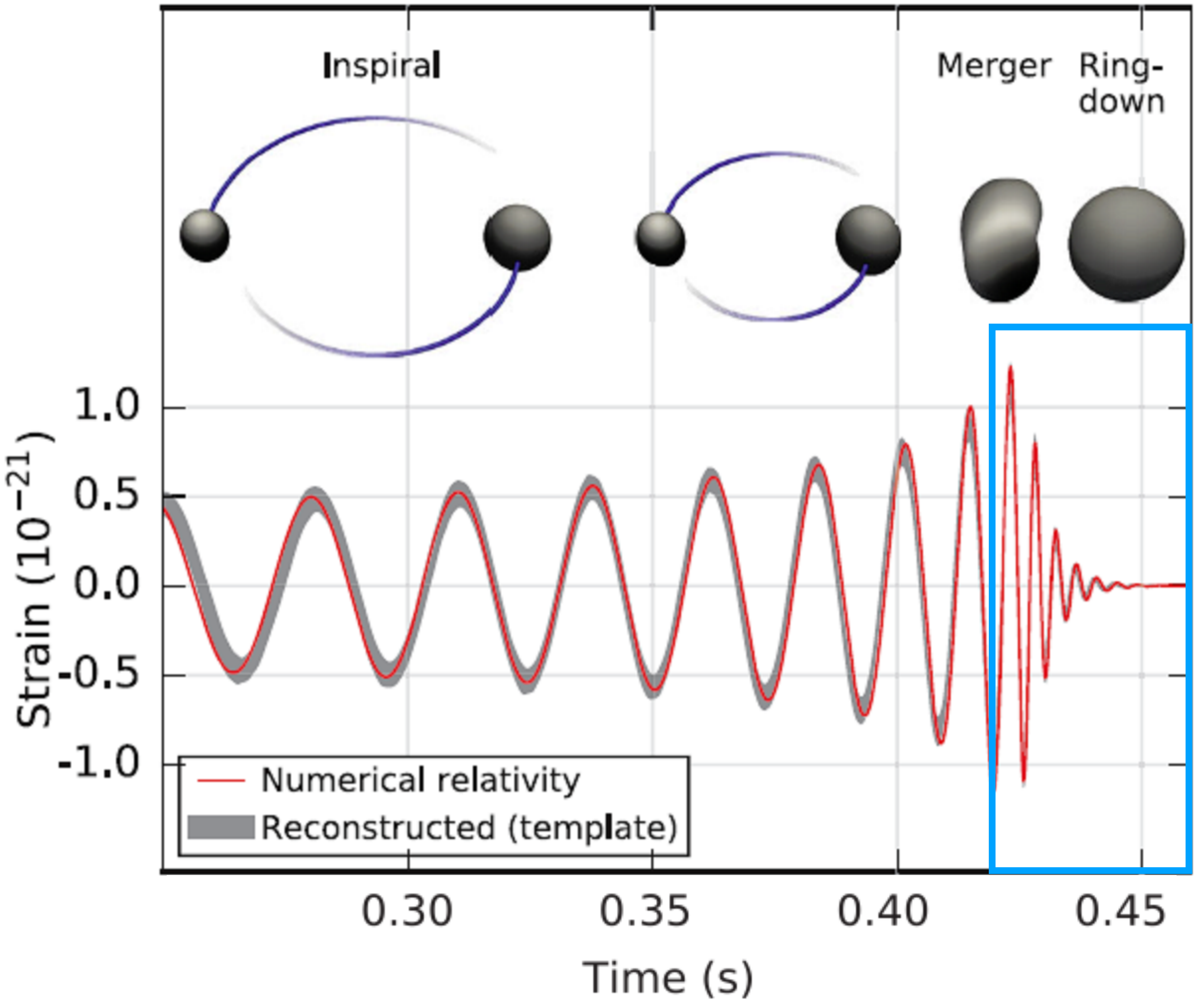


Image: Fig. 2 of LVC PRL **116**, 061102 (2016)

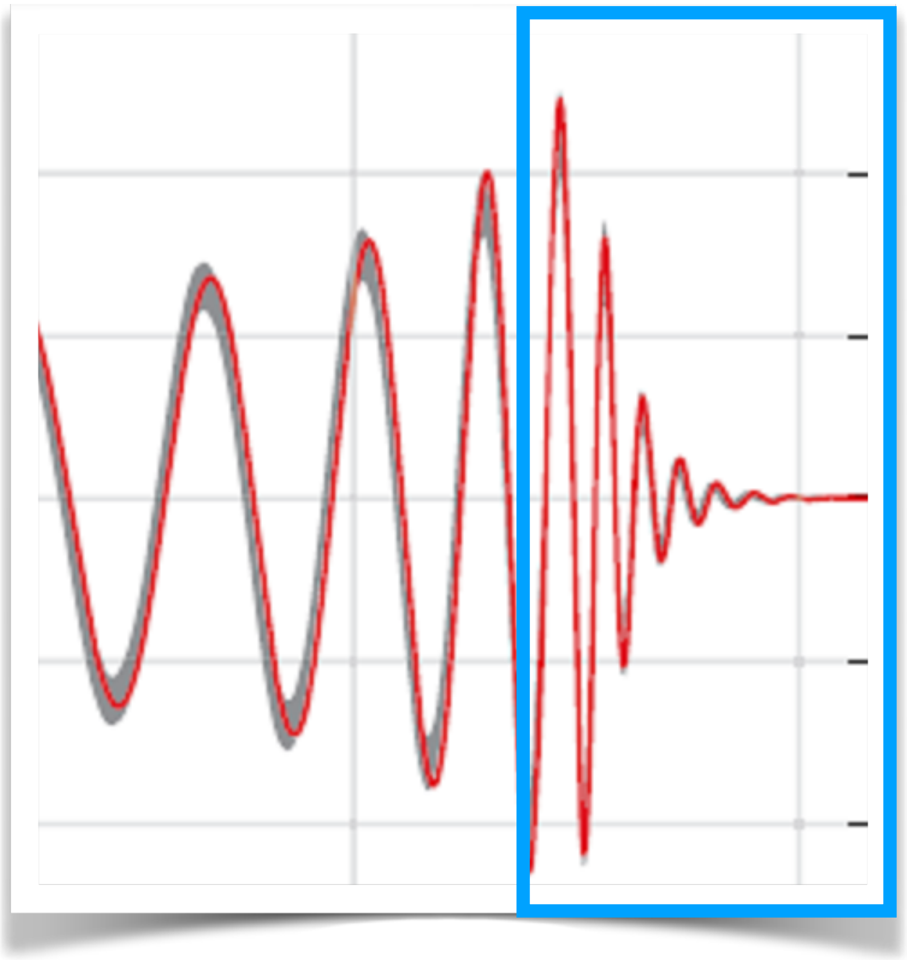
Quasinormal Modes (QNMs)

- Black holes radiate at certain QNM frequencies, $\omega_{\ell mn}$, fully determined by their mass M_{f} and angular momentum χ_{f}.
- QNMs are labelled by $\{\ell, m, n\}$ where $n = 0$ is the fundamental mode and $n > 0$ are overtones.
- Perturbation theory gives us this set of QNM frequencies.

Quasinormal Modes (QNMs)

A ringdown waveform is described by:

$$h^Q(u,\theta,\phi) = \sum_{\ell',m,n,\pm} \mathscr{A}^{\pm}_{\ell' mn} e^{-i\omega^{\pm}_{\ell' mn}(u-u_0)} {}_{-2}S_{\ell' m}(\theta,\phi;a\omega^{\pm}_{\ell' mn})$$



$\mathscr{A}^{\pm}_{\ell' mn}$ represents the QNM amplitudes

$\omega^{\pm}_{\ell' mn}$ are the QNM angular frequencies

${}_{-2}S_{\ell' m}(\theta,\phi;a\omega^{\pm}_{\ell' mn})$ are spin-weighted spheroidal harmonics

Quasinormal Modes (QNMs)

Spin-weighted spheroidal harmonics can be expanded in terms of the spin-weighted spherical harmonics as

$$ {}_sS_{\ell' m}(\theta, \phi; c) = \sum_{\ell'=\ell_{\min}, n} C_{\ell\ell' m}(c)\, {}_sY_{\ell m}(\theta, \phi) $$

such that our analytical model is

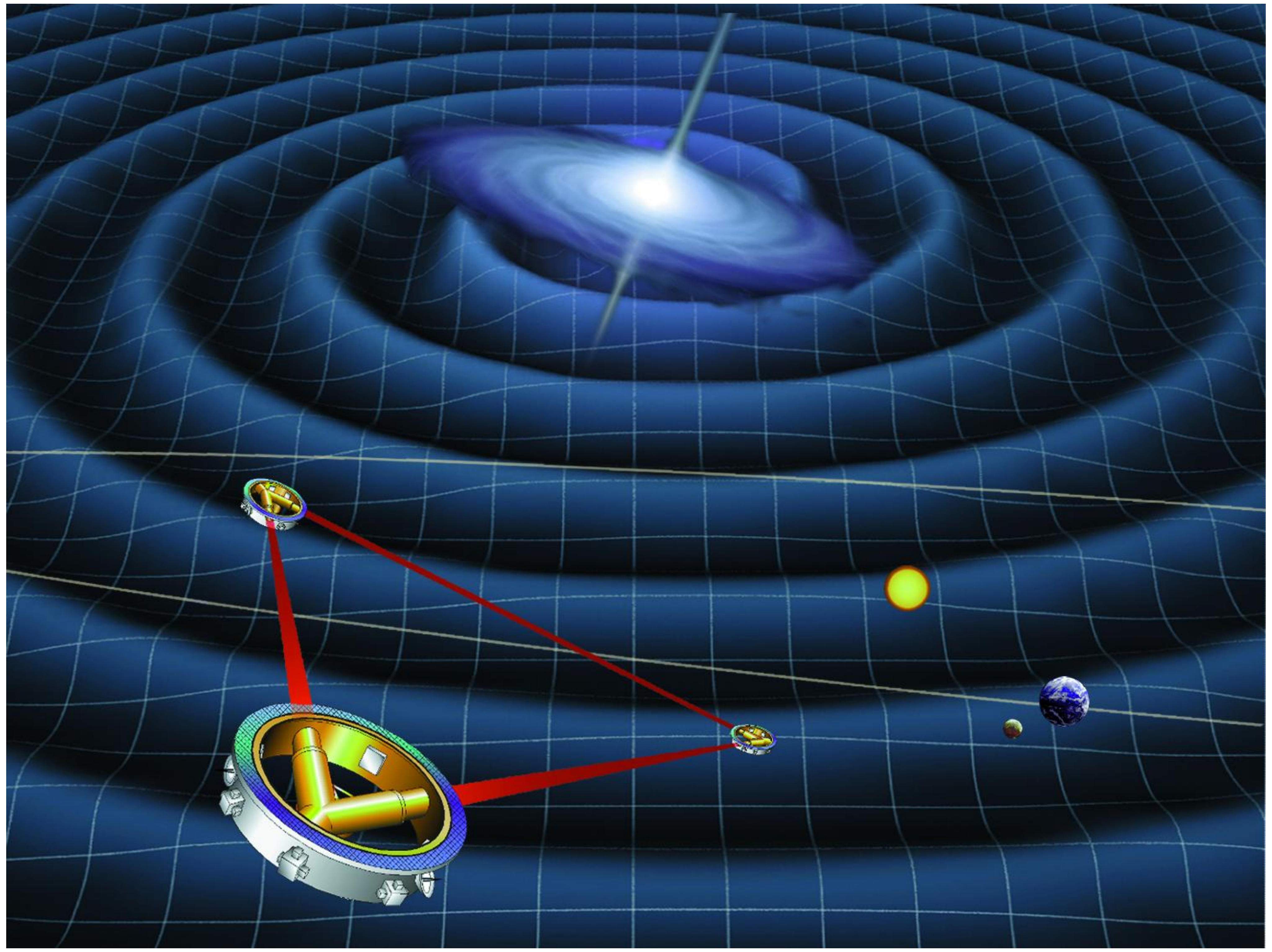
$$ h^Q(u, \theta, \phi) = \sum_{\ell', m, n, \pm} \left[\mathcal{A}^{\pm}_{\ell' mn} e^{-i\omega^{\pm}_{\ell' mn}(u-u_0)} \sum_{\ell} C_{\ell\ell' m}(a\omega^{\pm}_{\ell' mn})\, {}_{-2}Y_{\ell m}(\theta, \phi) \right] $$

Motivation

- The past decade has seen an increase in QNM exploration:
  - higher order modes [London+ PRD 2016, Baibhav+Berti PRD 2019, Cook PRD 2020]
  - retrograde modes [Dhani PRD 2021, Finch+Moore PRD 2021, LMZ+ PRD 2022]
  - overtones [London+ PRD 2014, Dhani+Sathyaprakash 2107.14195, Giesler+ PRX 2019]
  - quadratic modes [Cheung+ PRL 2023, Mitman+ PRL 2023, Cheung+ PRD 2024]
  - no-hair tests [Kamaretsos+ 2011, Isi+ PRL 2019, Dey+ PRD 2023, Pitte+ PRD 2024]
- Additional effects such as frame fixing and memory have also become important venues to understand ringdown. [Mitman+ PRD 2021, LMZ+ PRD 2022]

Motivation

- Ringdown provides us with a clean way to test GR
- Improved ringdown modelling gives more accurate parameter estimation for detections
- Crucial for next generation detectors that will capture a rich QNM spectrum



Numerical Recipe

- Using qnm, we solve for the QNM frequencies and spherical-spheroidal mixing coefficients. [Teukolsky 1974, Stein JOSS 2019]
  - Leaver's method for the radial sector [Leaver PRS 1985]
  - Spectral Eigenvalue method for the angular sector [Cook+Zalutskiy PRD 2014]
  - This allows us to fit for the QNM amplitudes for a given NR waveform.

$$h^Q(u,\theta,\phi) = \sum_{\ell',m,n,\pm} \left[\mathscr{A}^{\pm}_{\ell' mn} e^{-i\omega^{\pm}_{\ell' mn}(u-u_0)} \sum_{\ell} C_{\ell\ell' m}(a\omega^{\pm}_{\ell' mn})\, {}_{-2}Y_{\ell m}(\theta,\phi) \right]$$

Numerical Recipe

retrograde modes

spherical-spheroidal mixing

overtones

$$h^Q(u,\theta,\phi) = \sum_{\ell',m,n,\pm} \left[\mathscr{A}^{\pm}_{\ell' mn} e^{-i\omega^{\pm}_{\ell' mn}(u-u_0)} \sum_{\ell} C_{\ell\ell' m}(a\omega^{\pm}_{\ell' mn})\, {}_{-2}Y_{\ell m}(\theta,\phi) \right]$$

fit multiple modes

fix the BMS frame

L. London, D. Shoemaker, and J. Healy, PRD (2014)
E. Berti and A. Klein, PRD (2014)
V. Baibhav and E. Berti, PRD (2019)
M. Giesler et al., PRX (2019)

G.B. Cook , PRD 102 (2020)
A. Dhani and B.S. Satyaprakash, arXiv: 2107.14195 (2021)
X. Li et al,. PRD 105 (2022)
K. Mitman et al., PRD 104 (2021)
L. Magaña Zertuche et al., PRD 105 (2022)

Extracting QNM Amplitudes

Rank (ℓ, m) modes by power

Pick loudest mode with lowest n and add both prograde and retrograde modes $(\ell, m, n, \pm)$

Fit modes by minimization of residual power:
$$P(h) = \langle h^{NR} - h^{Q} \,|\, h^{NR} - h^{Q} \rangle$$

$$\langle a, b \rangle \equiv \int_{u_0}^{u_f} du \int_{S^2} d\Omega \, a^*(u, \theta, \phi) b(u, \theta, \phi)$$

Build analytical waveform h^Q

Find fraction of unmodeled power:
$$\mathscr{F}(h) = \frac{\langle h^{NR} - h^{Q} \,|\, h^{NR} - h^{Q} \rangle}{\langle h^{NR} \,|\, h^{NR} \rangle}$$

Calculate the all mode mismatch:
$$\mathscr{M} = 1 - \frac{\langle h^{NR} \,|\, h^{Q} \rangle}{\sqrt{\langle h^{NR} \,|\, h^{NR} \rangle \langle h^{Q} \,|\, h^{Q} \rangle}}$$

BMS Transformations

- Bondi-Metzner-Sachs (BMS) group is the symmetry group for asymptotically flat spacetimes at null infinity described as Lorentz $\ltimes$ supertranslations.
- Supertranslations are an infinite-dimensional group of transformations that contain spacetime translations as a subgroup.
- BH remnants in simulations are supertranslated with respect to the remnants described by Teukolsky's equations.

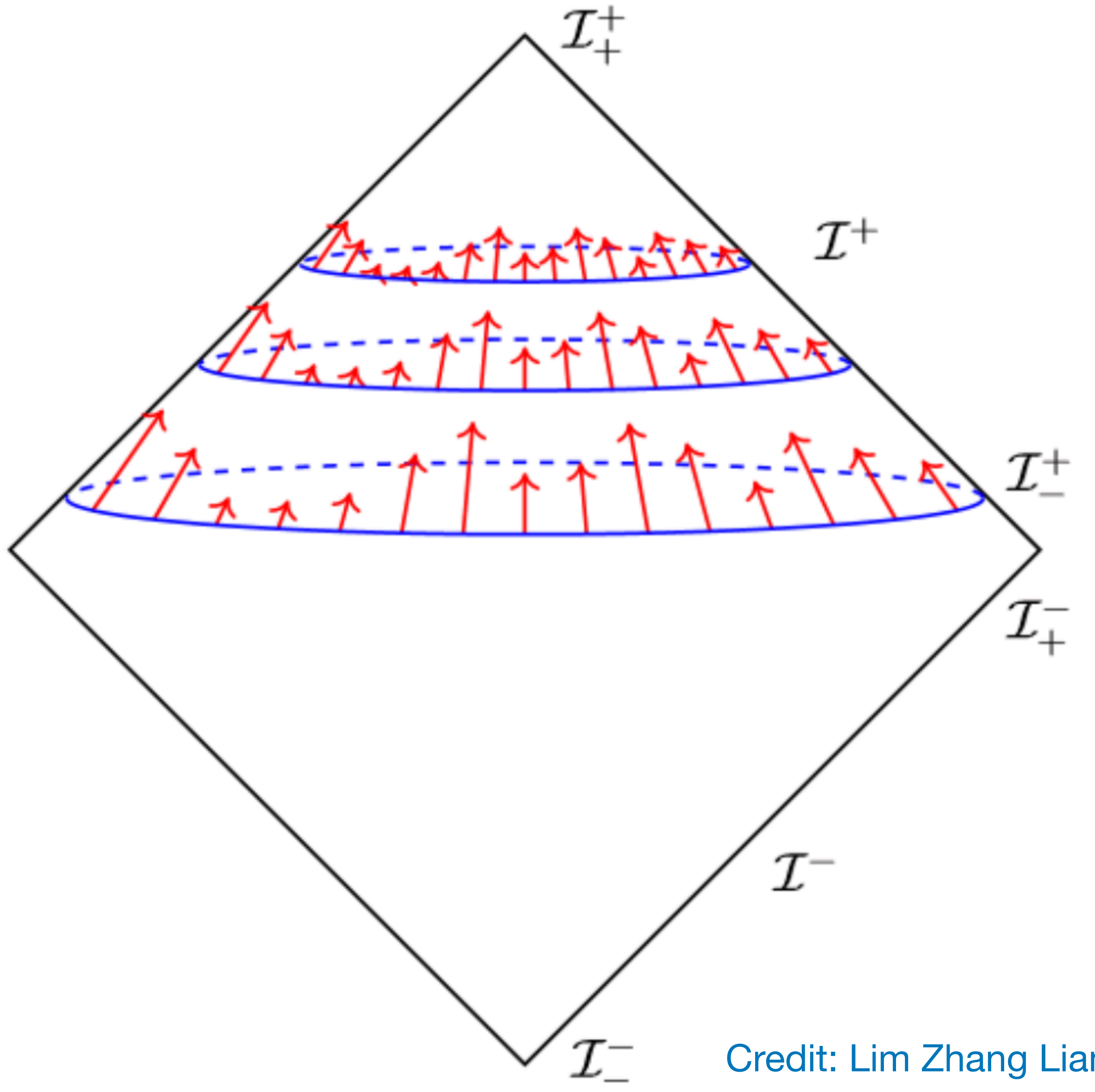
BMS Transformations

These supertranslations are angle-dependent time translations such that

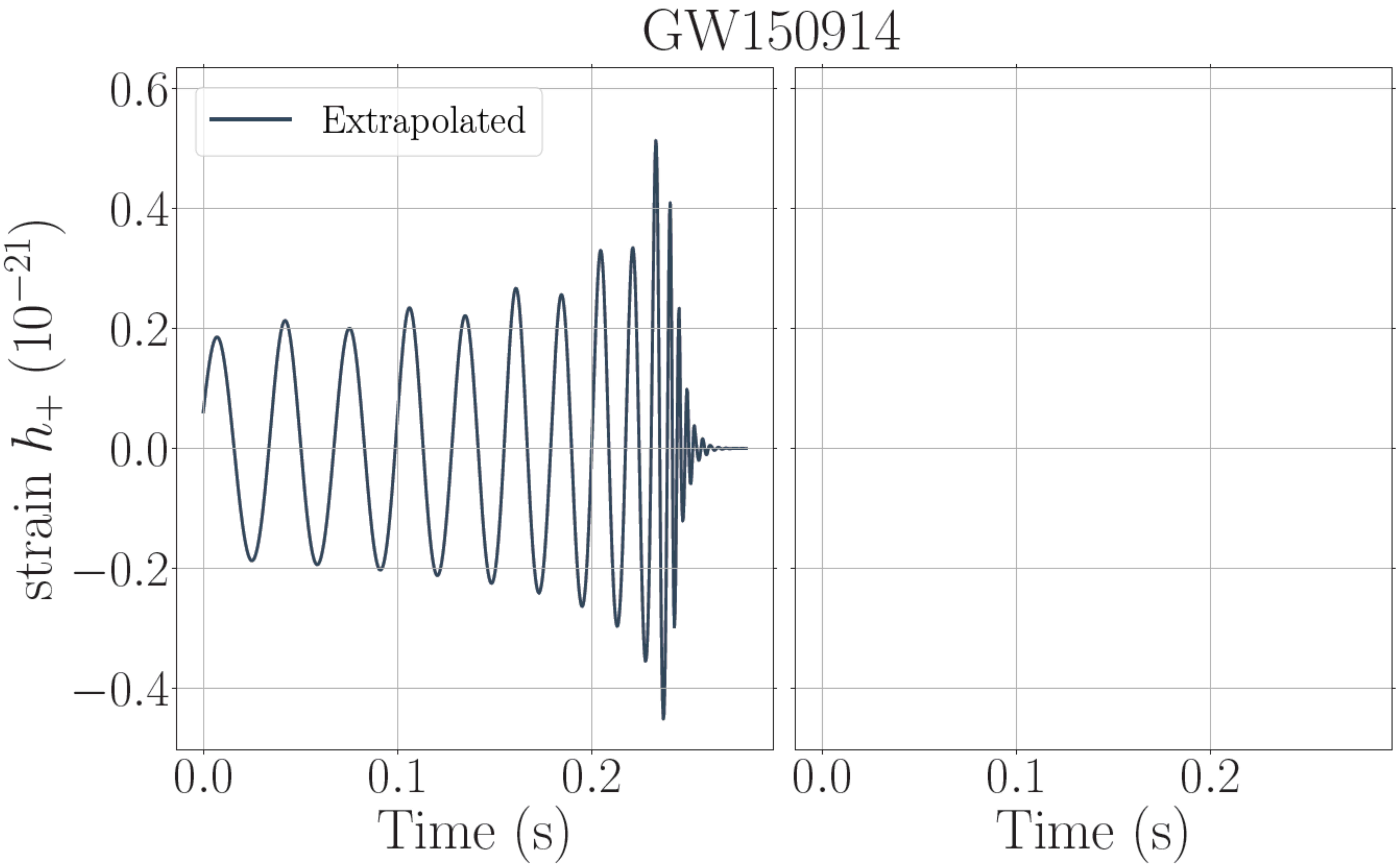
$$u' = u - \alpha(\theta, \phi)$$

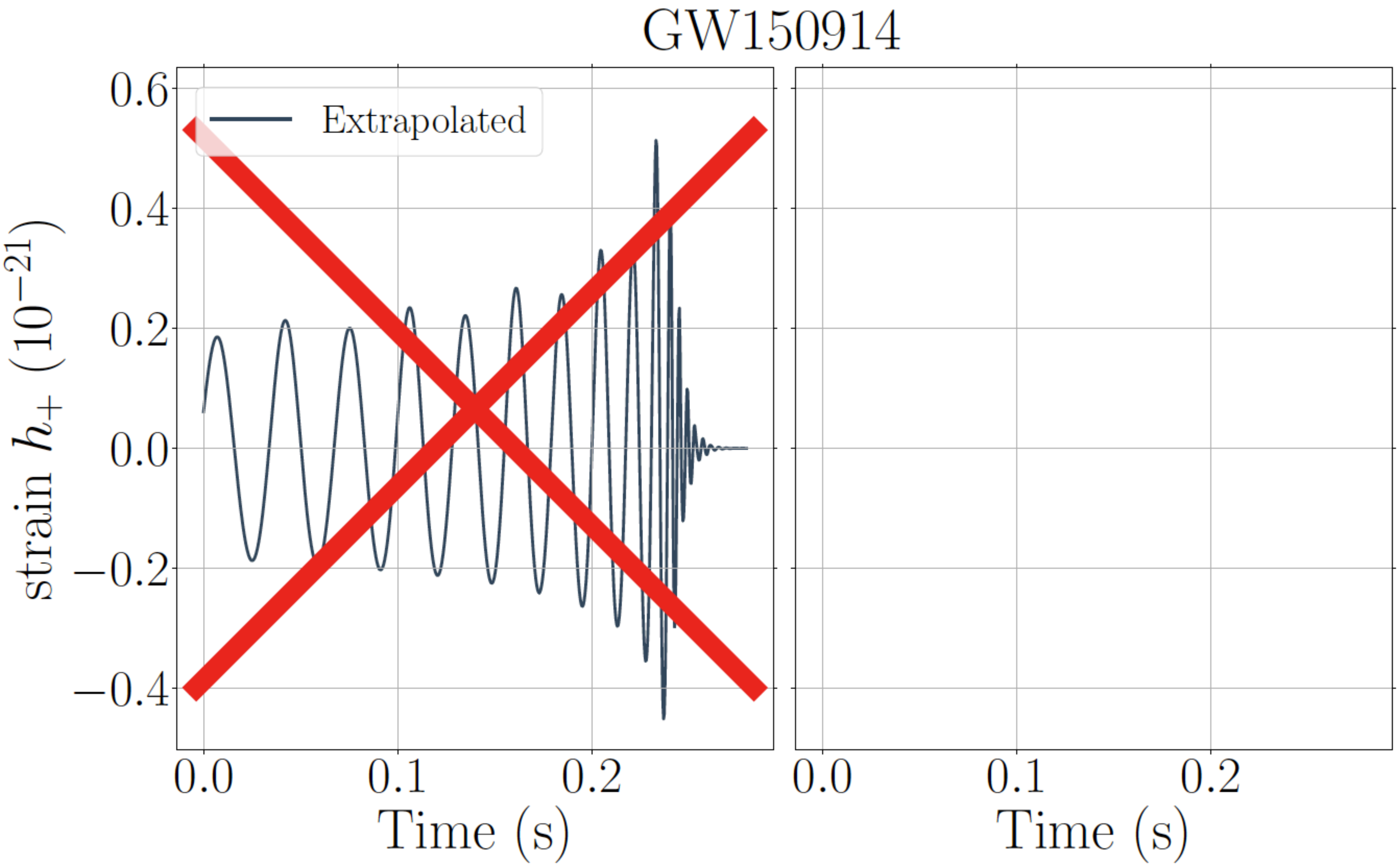
and the strain becomes

$$h'(u', \theta, \phi) = \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!} \left(-\alpha(\theta, \phi) \frac{\partial}{\partial u} \right)^k h(u, \theta, \phi)}_{\textbf{mode mixing}} - \underbrace{\bar{\eth}^2 \alpha(\theta, \phi)}_{\textbf{constant shift}}$$



Credit: Lim Zhang Liam


GW150914
Extrapolated
strain h_+ (10^{-21})
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.1
0.2
Time (s)
0.0
0.1
0.2
Time (s)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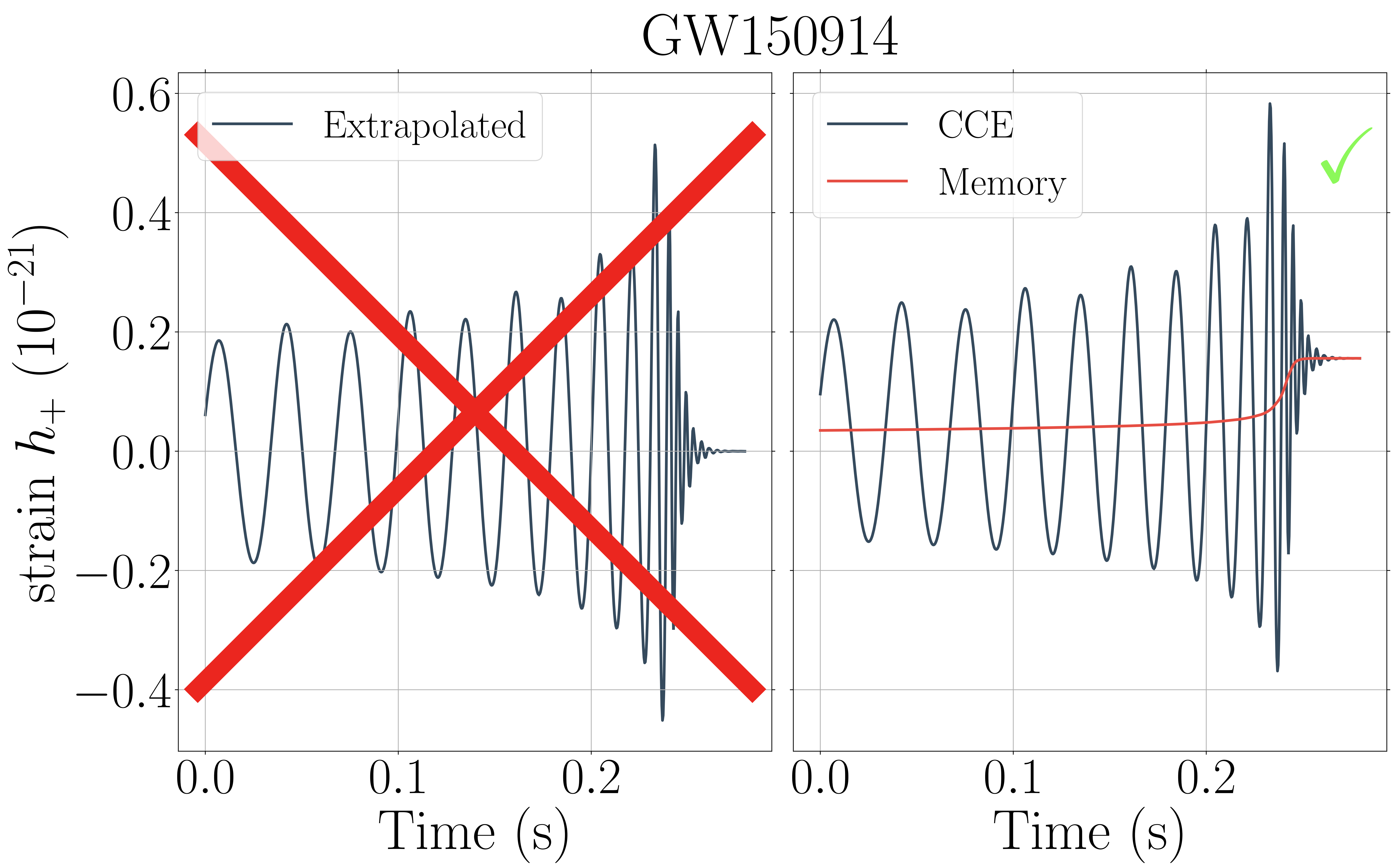
GW150914
strain h_+ (10^{-21})
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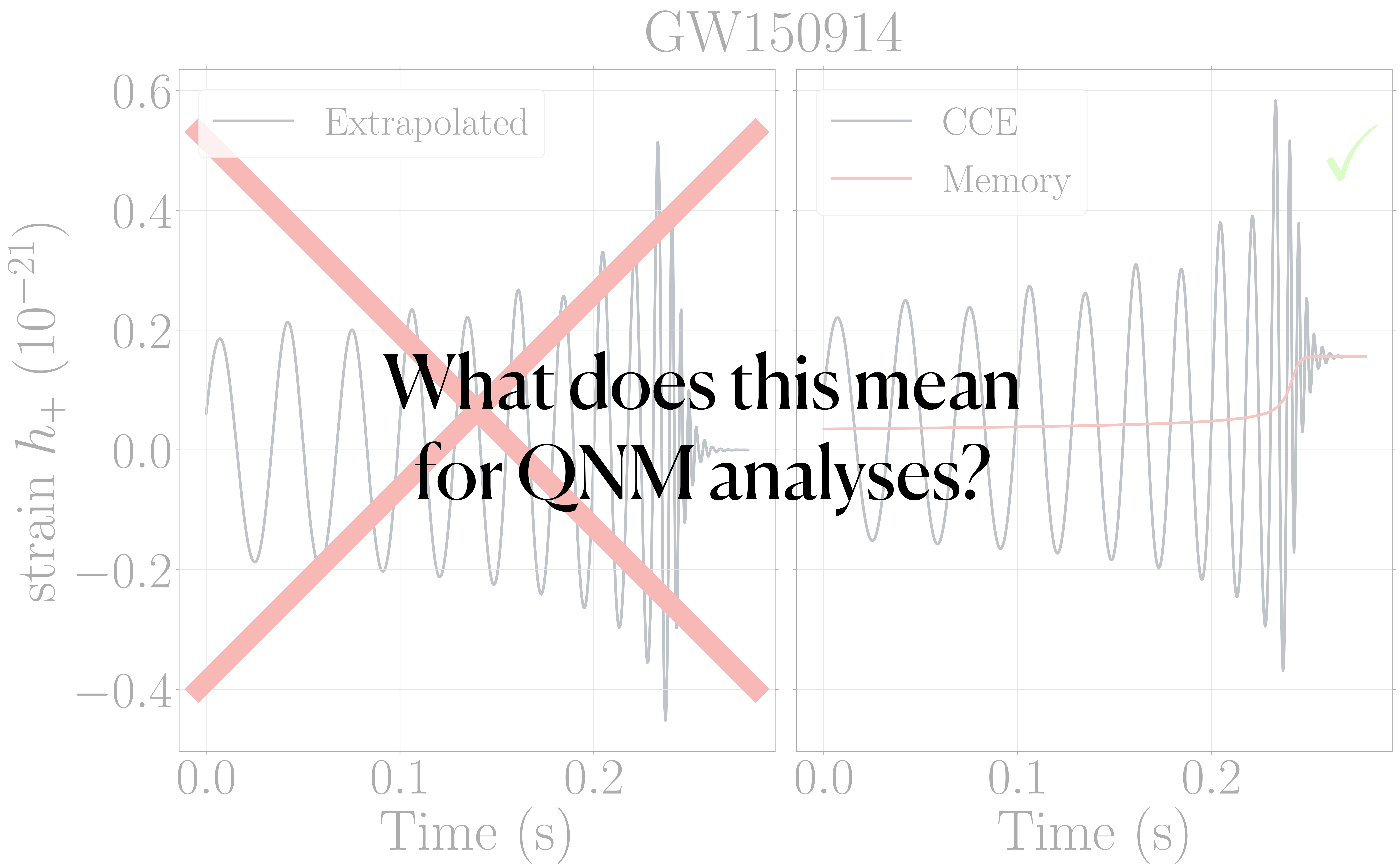
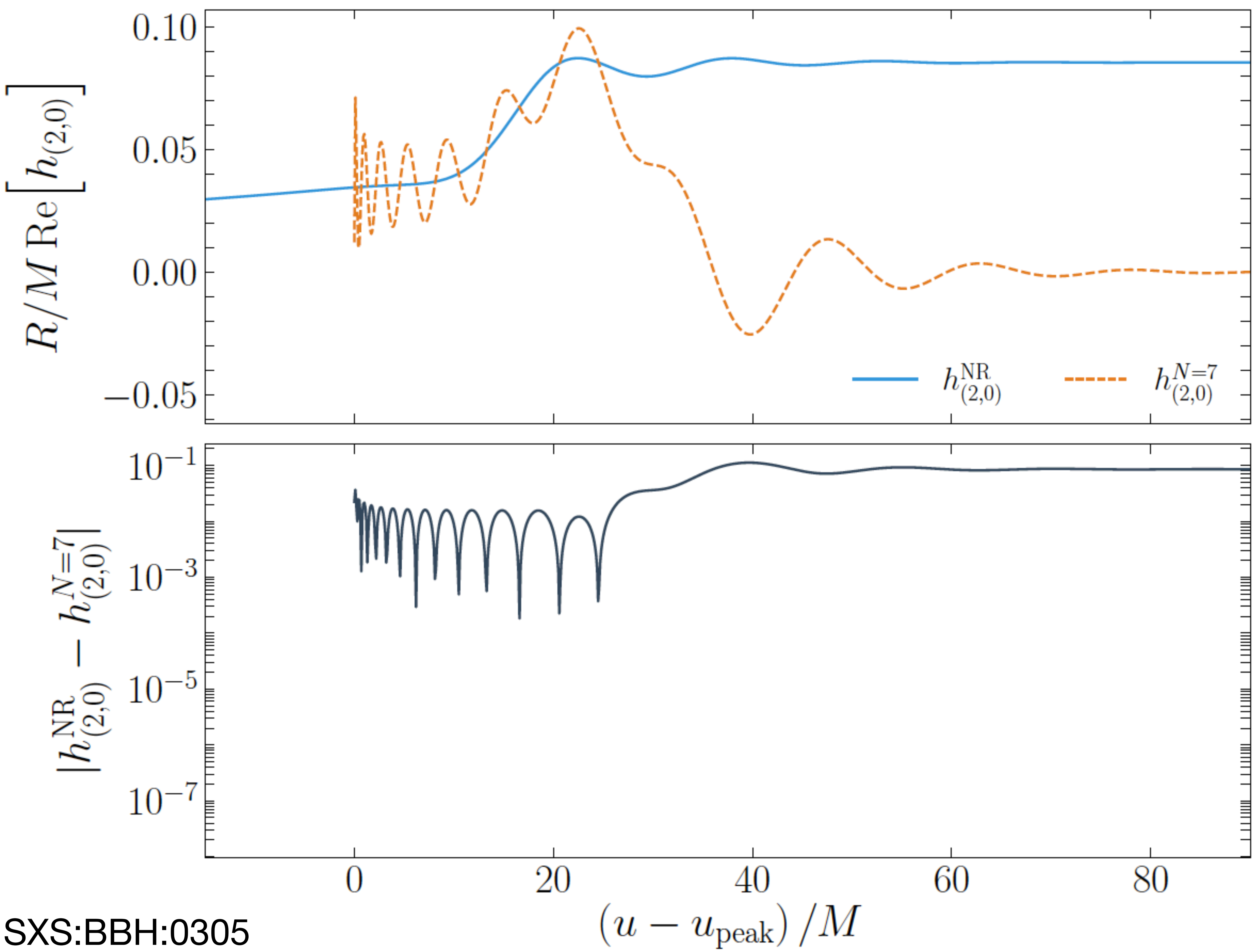
CCE
Memory
0.0
0.1
0.2
Time (s)
Time (s)



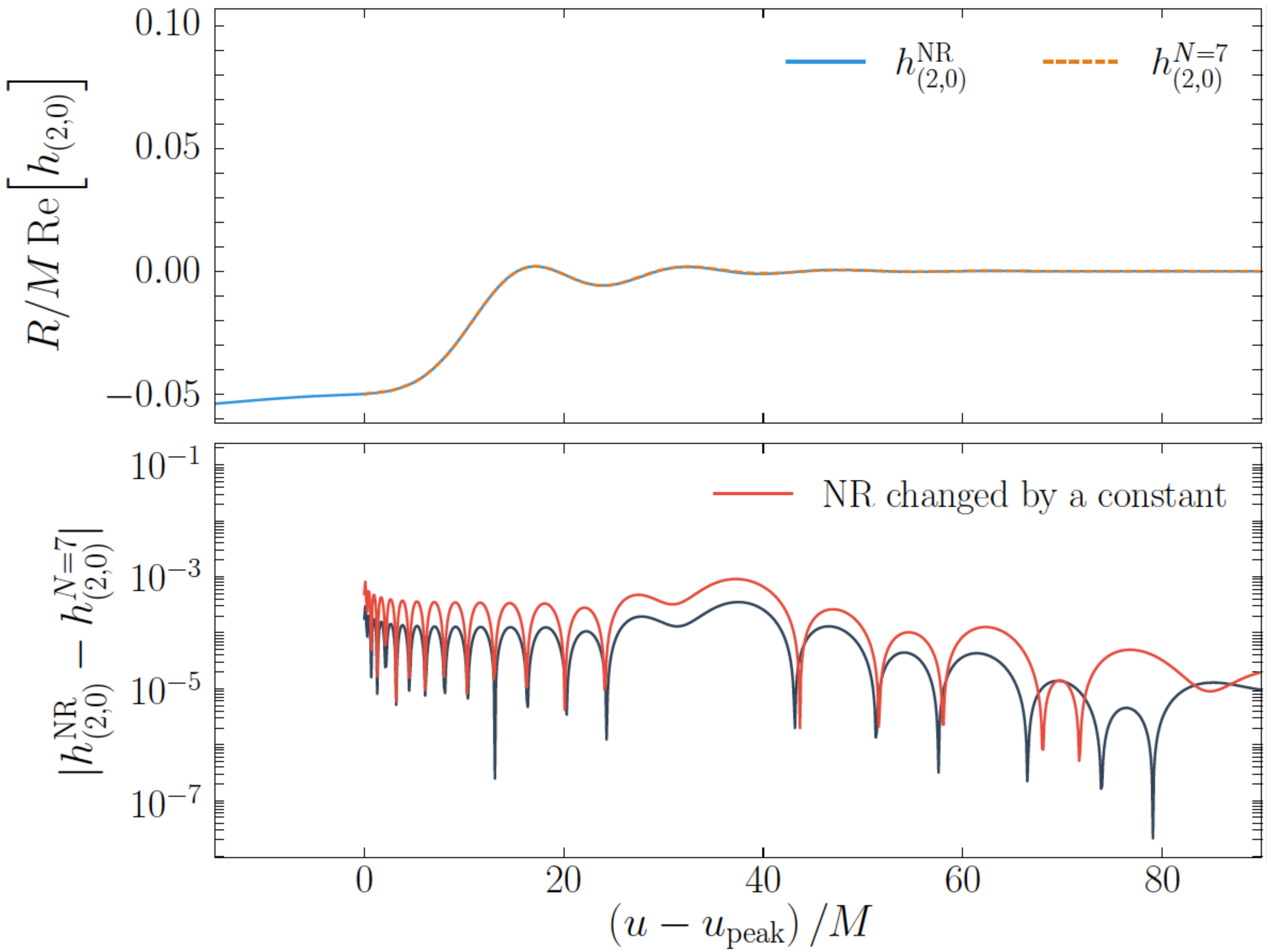
What does this mean for QNM analyses?

Importance of Mapping to the Superrest Frame



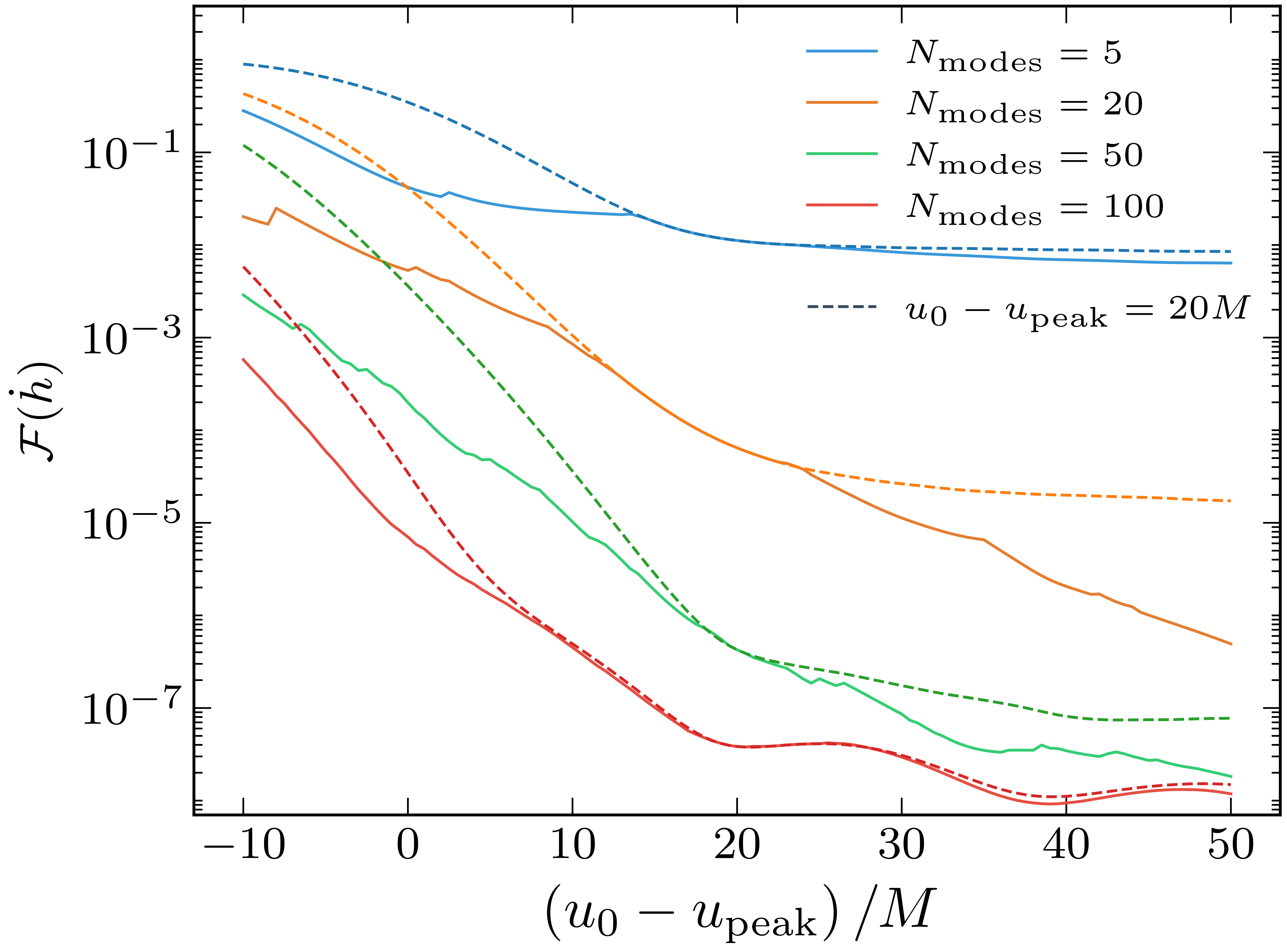
SXS:BBH:0305

Importance of Mapping to the Superrest Frame



Using multimode and BMS frame fixing together...

Importance of Fitting Multiple Modes



How much power is modeled?

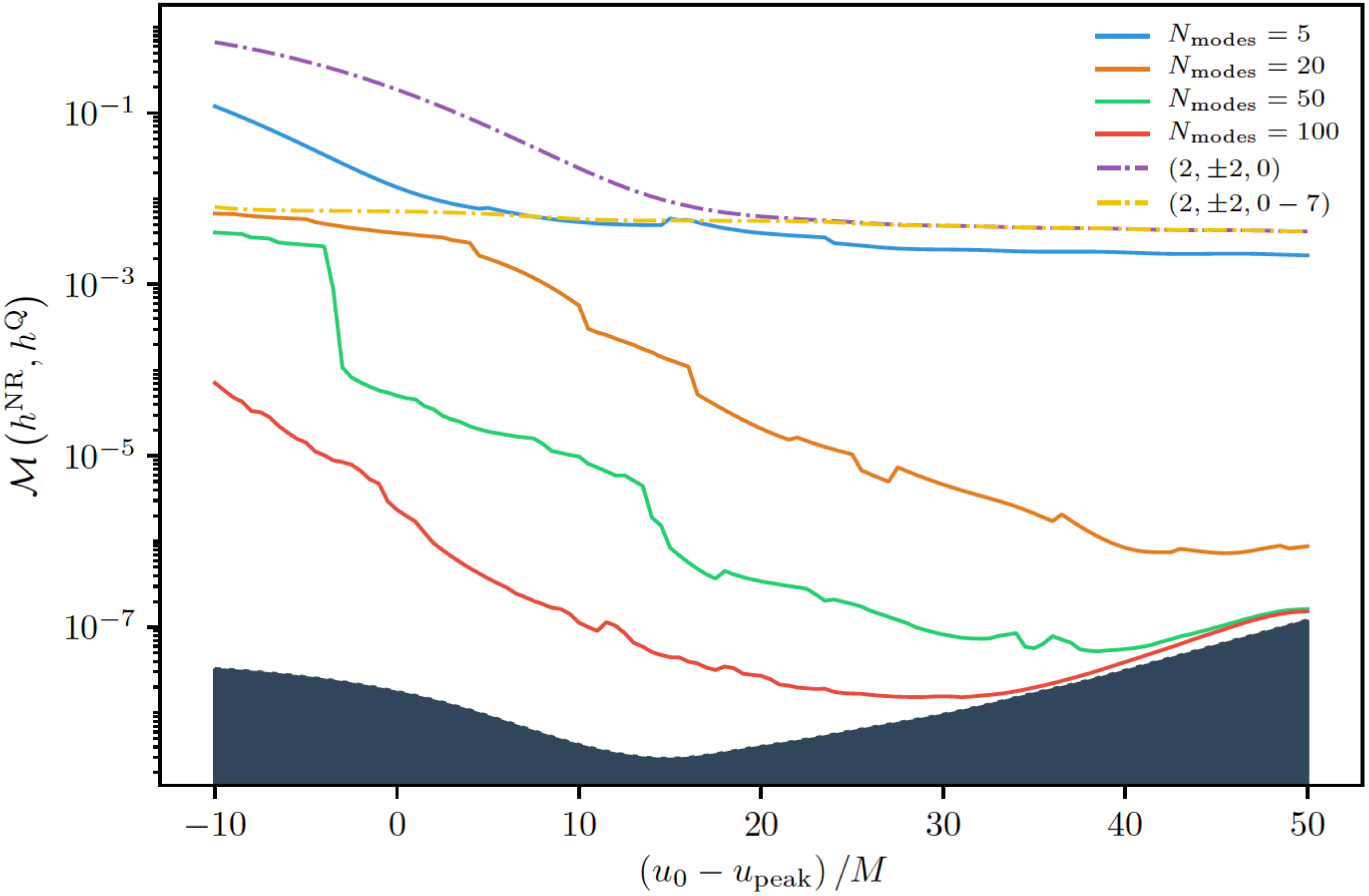
For SXS:BBH:0305

N=5: 98%

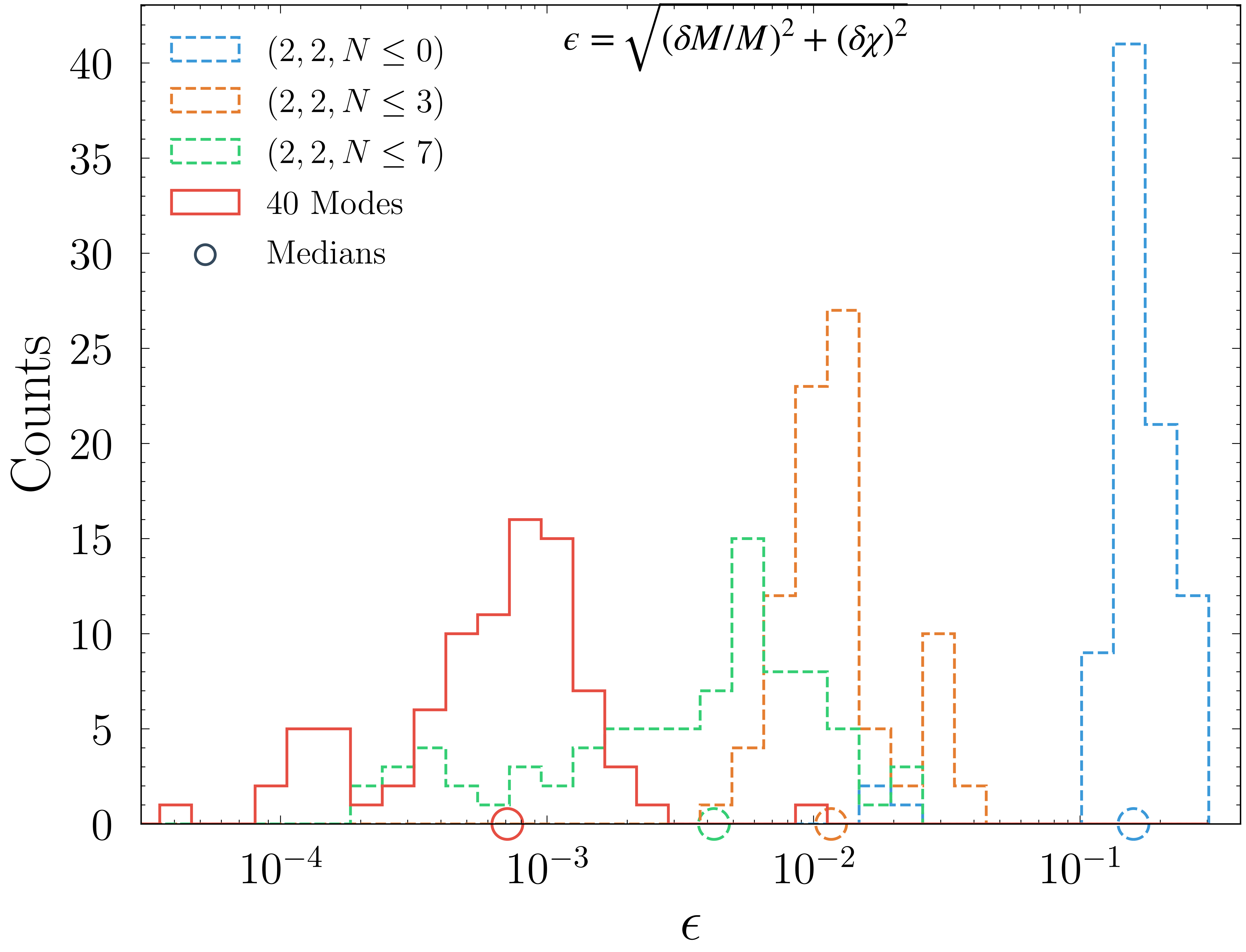
N=20: 99%

N=100: 99.999%

Importance of Fitting Multiple Modes



Importance of Fitting Multiple Modes



86 simulations from NRHyb3dq8

Extracting QNM Amplitudes

	qnmfits	**KerrRingdown**
Language	Python	Mathematica
Fit quantities	QNM amplitudes, final mass and spin	
Fit type	Linear least-squares	Linear least-squares, mode-limited eigenvalue**
Multimode?	✓	✓
Greedy algorithm?	✓	✗
Minimization of	Power residual*	Power residual or Mismatch
Waveform type	Any, easy loading of CCE waveforms	

github.com/sxs-collaboration/qnmfits

qnmfits

github.com/cookgb/KerrRingdown

KerrRingdown

*LMZ+ PRD 2021 **Cook PRD 2020

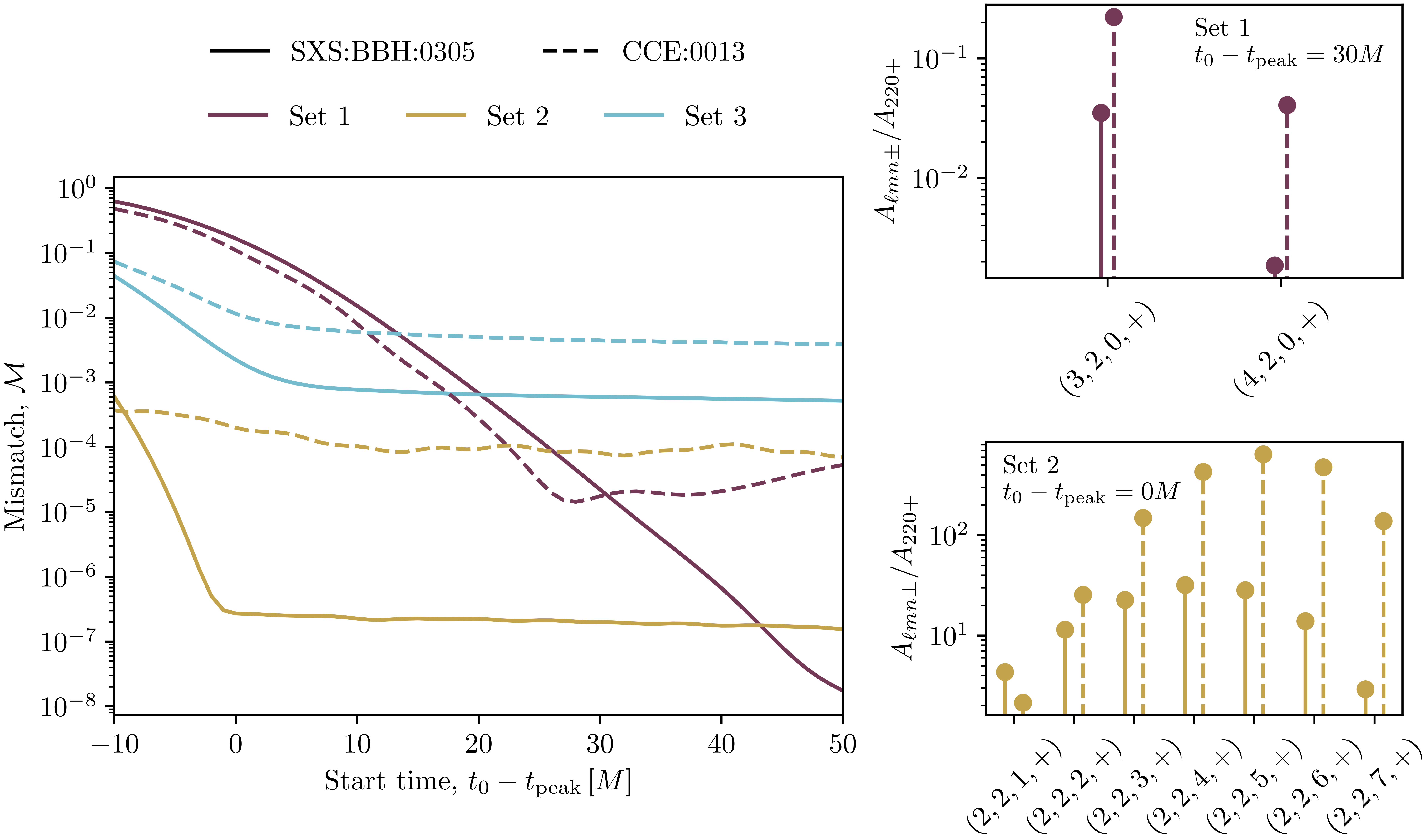
LMZ, Gao, Finch, and Cook 2502.03155

qnmfits Demonstration

Set 1: $(\ell, 2, 0, \pm)$ mode mixing for $\ell = 2 - 4$

Set 2: $(2, 2, n, +)$ overtones up to $n = 7$

Set 3: Top 20 modes

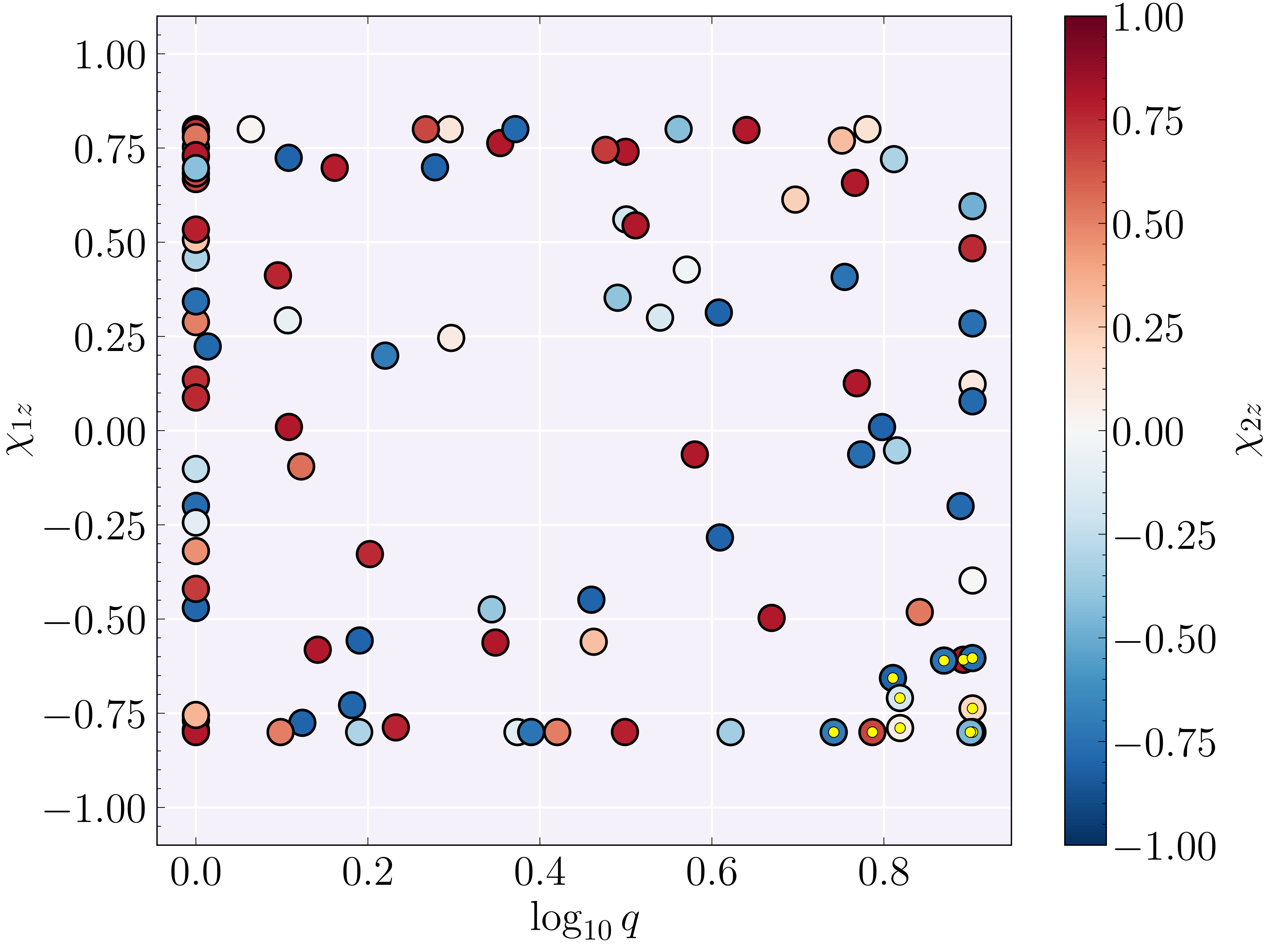


Don’t have a waveform but have initial masses and spins?

Don't have a waveform but have initial masses and spins?

Surrogate Model: NRSur3dq8_RD

- Surrogate modelling quickly gives us waveforms in parameter space not yet simulated.
- With `qnmfits` and NR waveforms, we build surrogate model `NRSur3dq8_RD` [LMZ+ 2408.05300]
- We use 102 CCE waveforms in the superrest frame of the remnant.

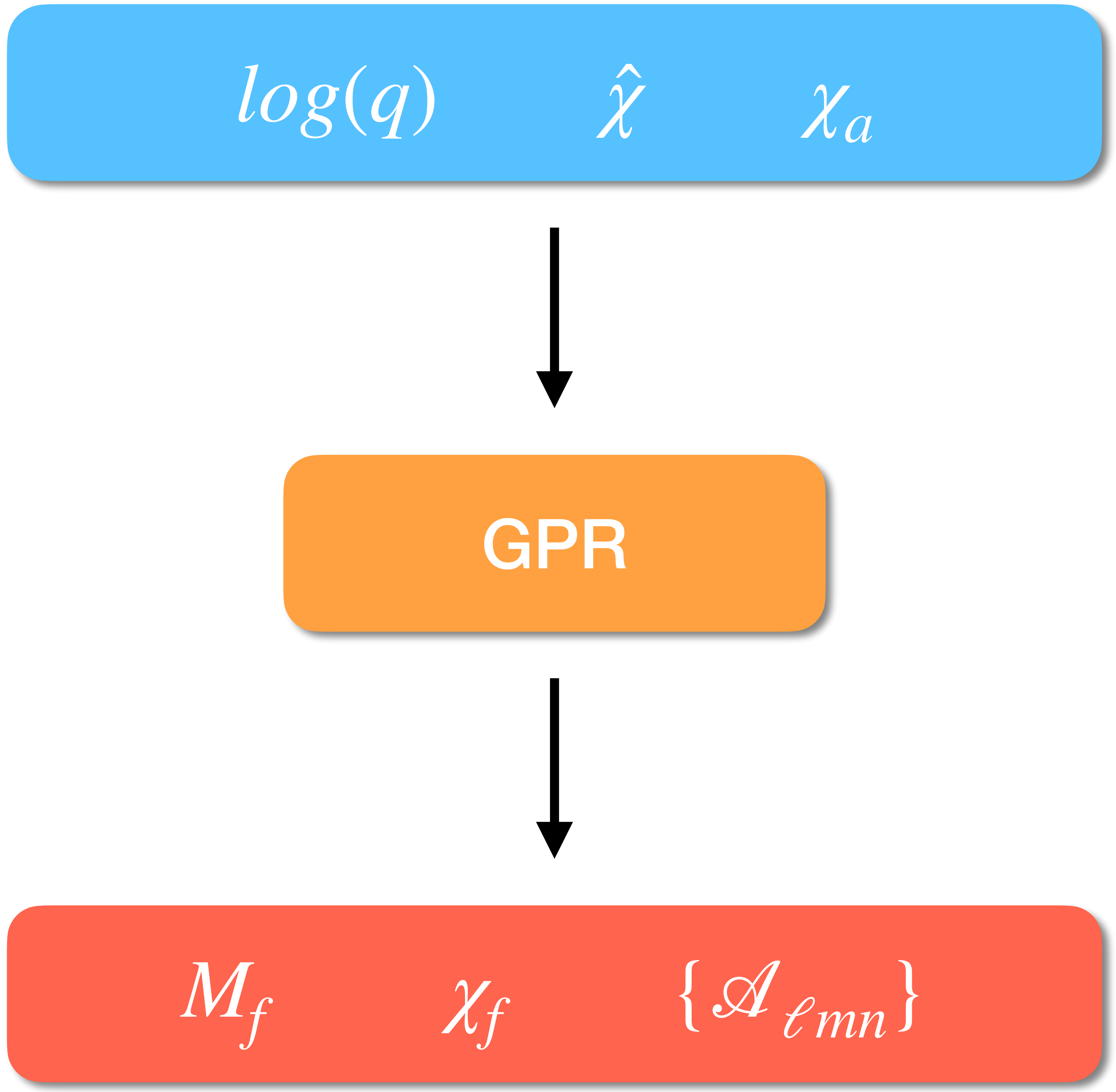


Surrogate Model: NRSur3dq8_RD

- Map BBH system parameters to the remnant parameters using Gaussian process regression (GPR).
- Make predictions by interpolating between simulation data.

$$\hat{\chi} = \frac{\chi_{\rm eff} - 38\eta(\chi_{1z} + \chi_{2z})/113}{1 - 76\eta/113}$$

$$\chi_a = \frac{1}{2}(\chi_{1z} - \chi_{2z})$$



QNM Fitting Challenges

How many modes do we need to fit?

QNM Fitting Challenges

How many modes do we need to fit?

Which modes do we pick?

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QNM Fitting Challenges

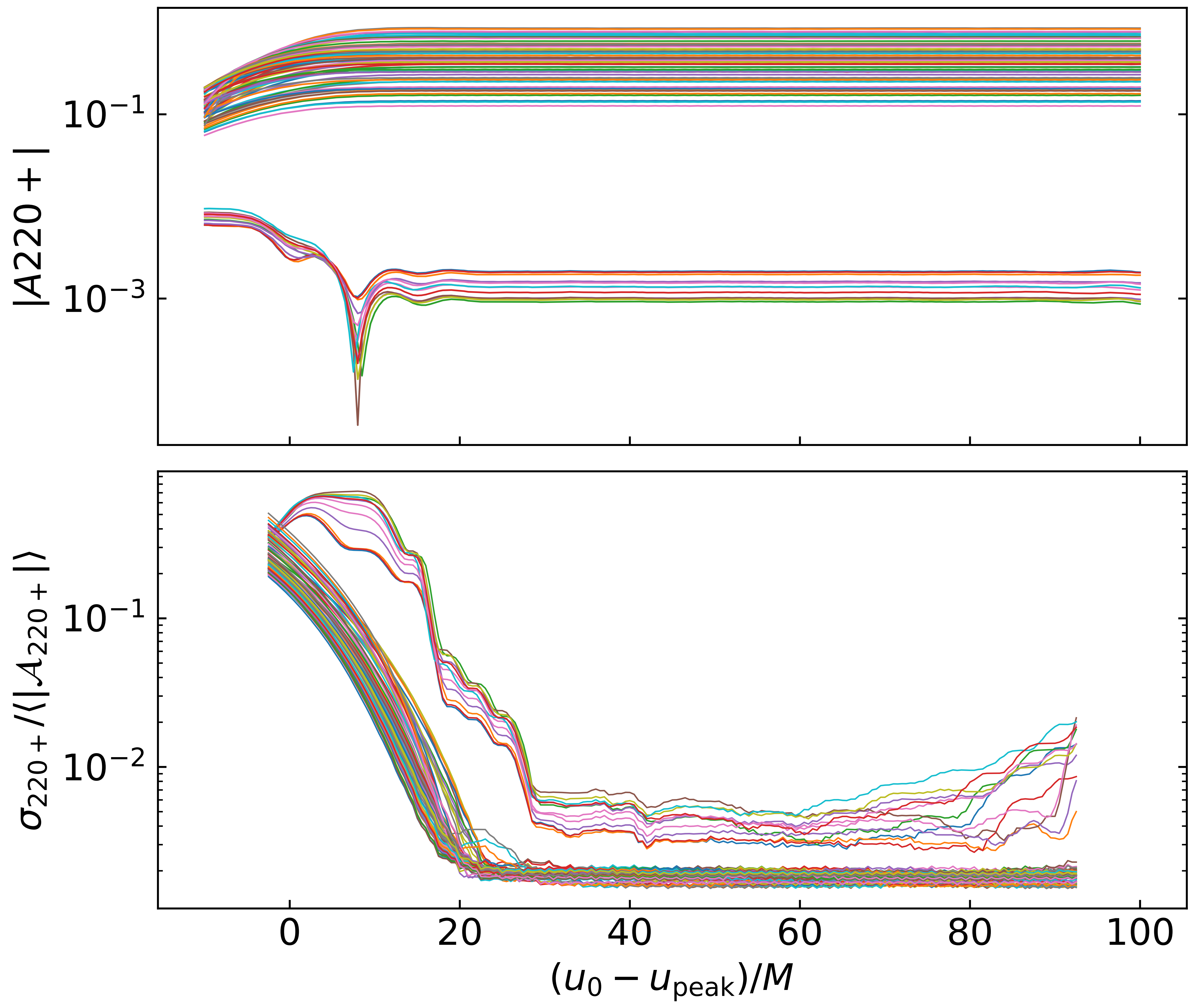
How many modes do we need to fit?

Which modes do we pick?

At what time do we fit them?

Mode Stability in NRSur3dq8_RD

- Align the waveforms in time and phase.
- Fit 168 QNMs for all NR waveforms for $u_0 - u_{\text{peak}} \in [-10M, 50M]$
- Greedily pick the loudest 26 modes across simulations and fit them for $u_0 - u_{\text{peak}} \in [-10M, 100M]$
- Check mode stability by calculating the variance of the QNM amplitude over a time window normalized by the mean absolute amplitude.



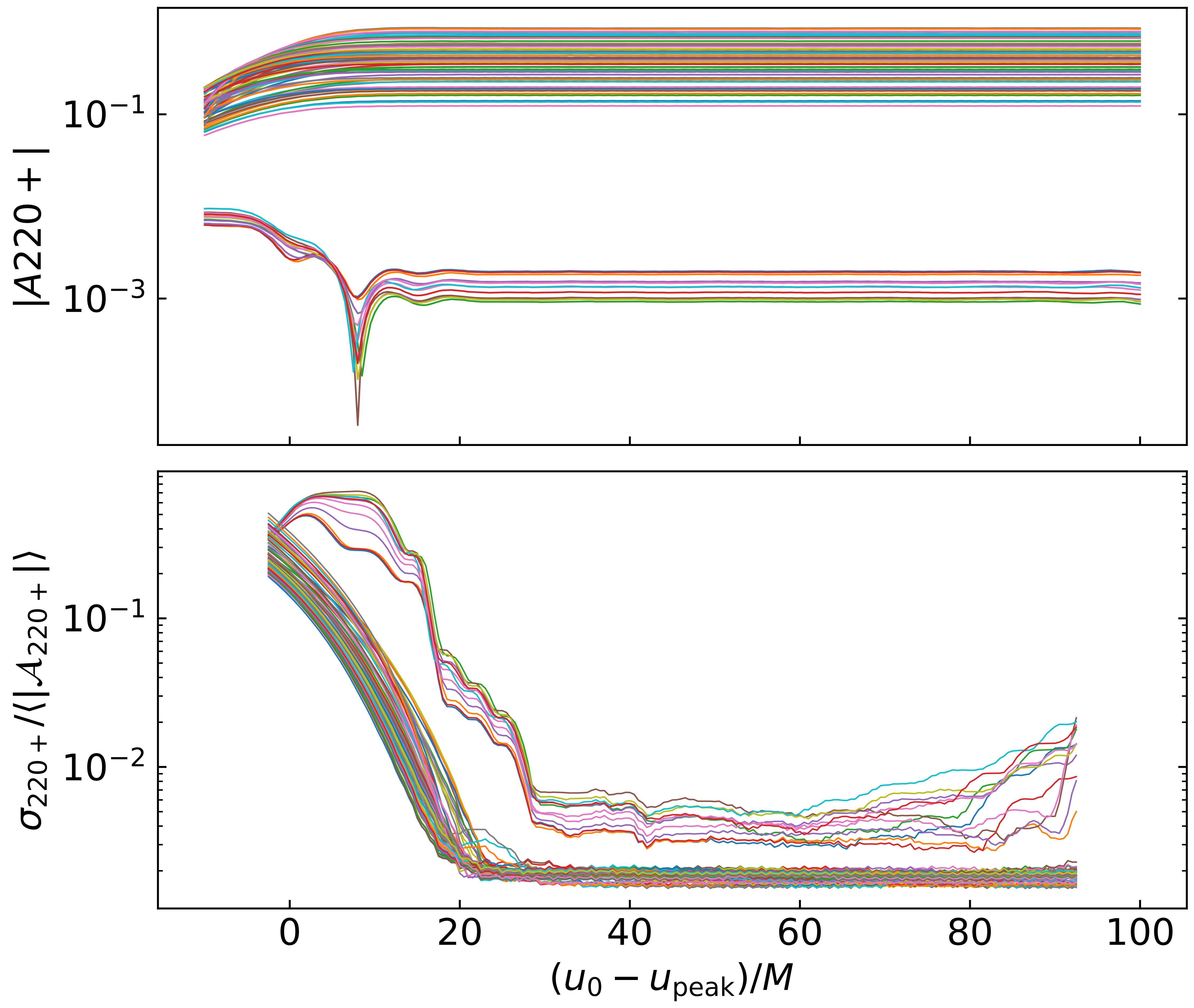
Mode Stability in NRSur3dq8_RD

- Keep only stable modes and choose a time when all modes vary minimally.
- Mode stability tells us to fit no earlier than $t_0 - t_{\text{peak}} = 20M$

Modes:

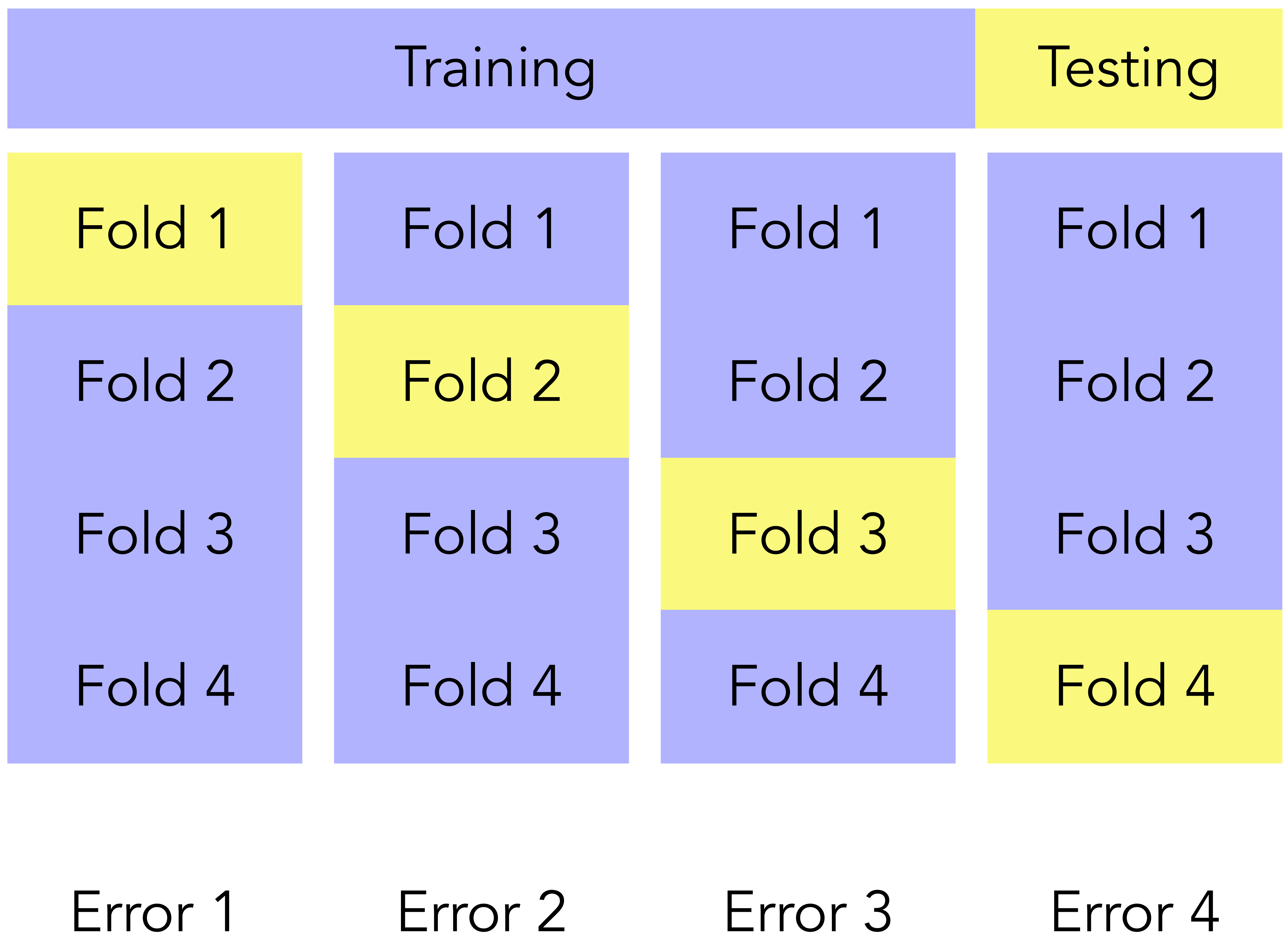
$$\mathscr{A}^+_{220},\ \mathscr{A}^+_{221},\ \mathscr{A}^+_{200},\ \mathscr{A}^+_{320},\ \mathscr{A}^+_{440}$$

$$\mathscr{A}^-_{2-20},\ \mathscr{A}^-_{2-21},\ \mathscr{A}^-_{200},\ \mathscr{A}^-_{3-20},\ \mathscr{A}^-_{4-40}$$



QNM Amplitude Errors in NRSur3dq8_RD

- Small number of waveforms for training requires k-fold cross-validation for predicting errors
- Helps avoid bias in the model
- Allows to train and test over full dataset

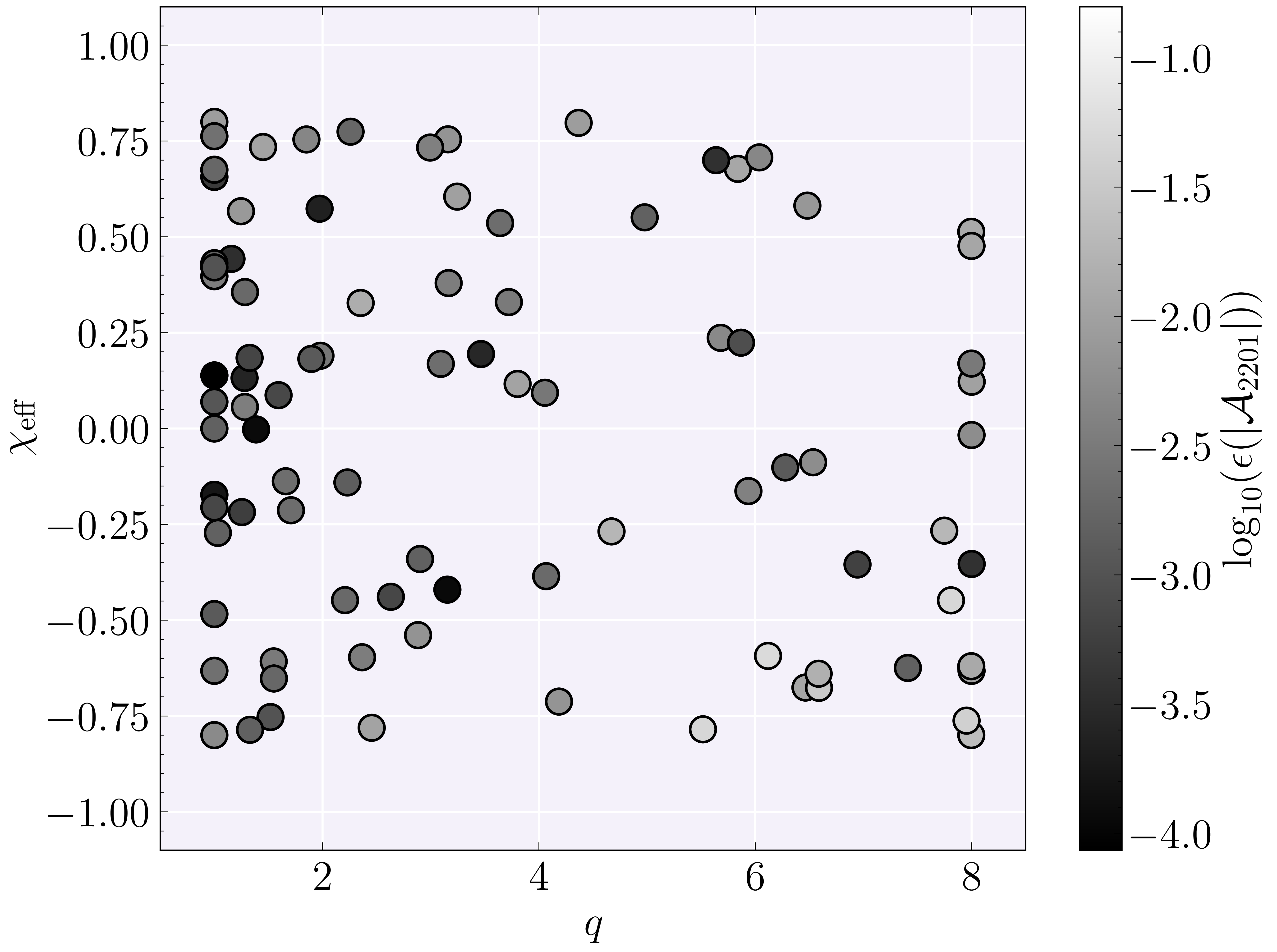


QNM Amplitude Errors in NRSur3dq8_RD

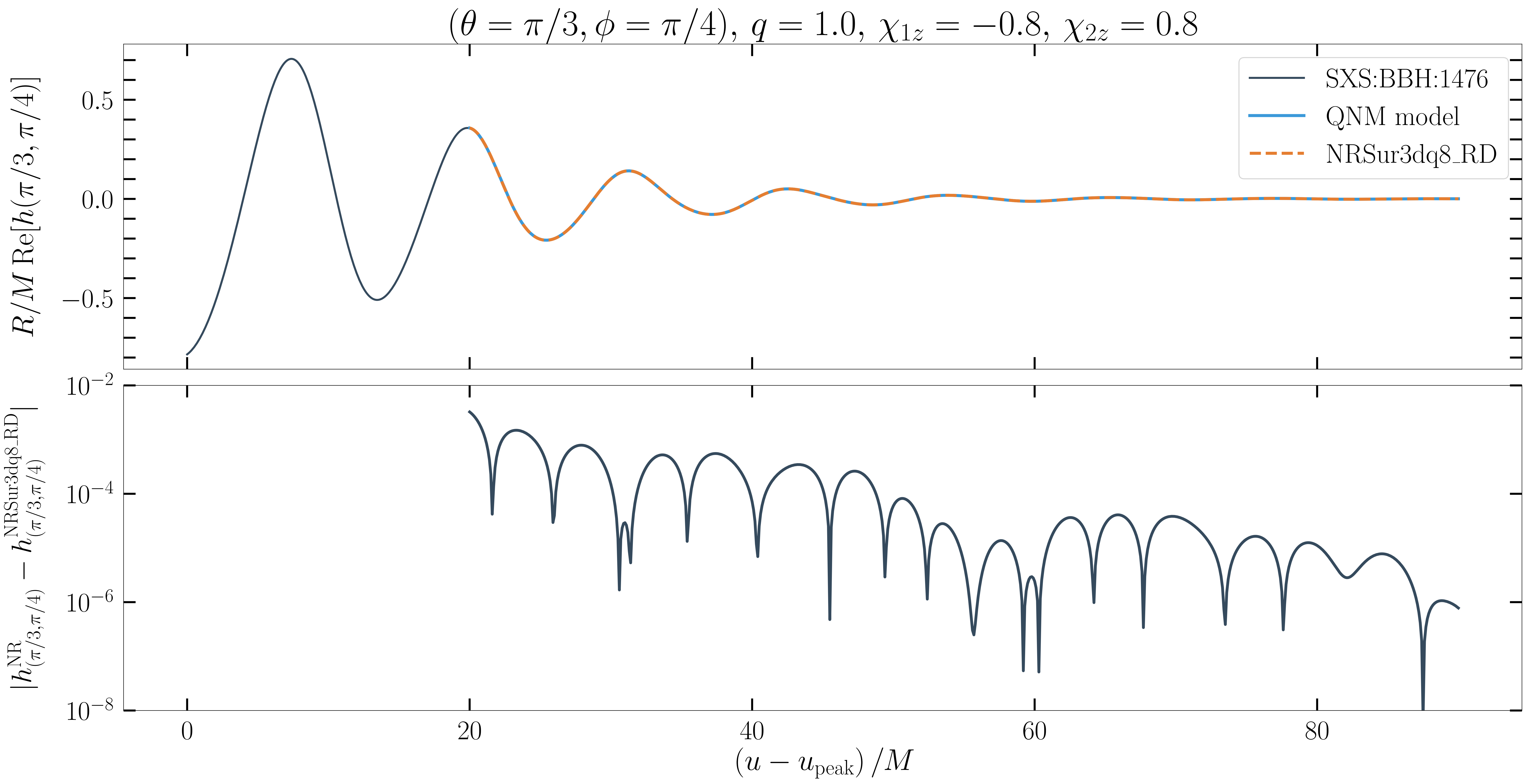
Median relative errors in QNMs

QNM	$\epsilon(\Re[\mathcal{A}_{\ell mn}])$	$\epsilon(\Im[\mathcal{A}_{\ell mn}])$	$\epsilon(\lvert\mathcal{A}_{\ell mn}\rvert)$
(2,2,0,+)	0.0020	0.061	0.0024
(2,2,1,+)	0.060	0.064	0.050
(2,0,0,+)	0.0026	0.0089	0.0037
(3,2,0,+)	0.024	0.012	0.0084
(4,4,0,+)	0.035	0.011	0.011

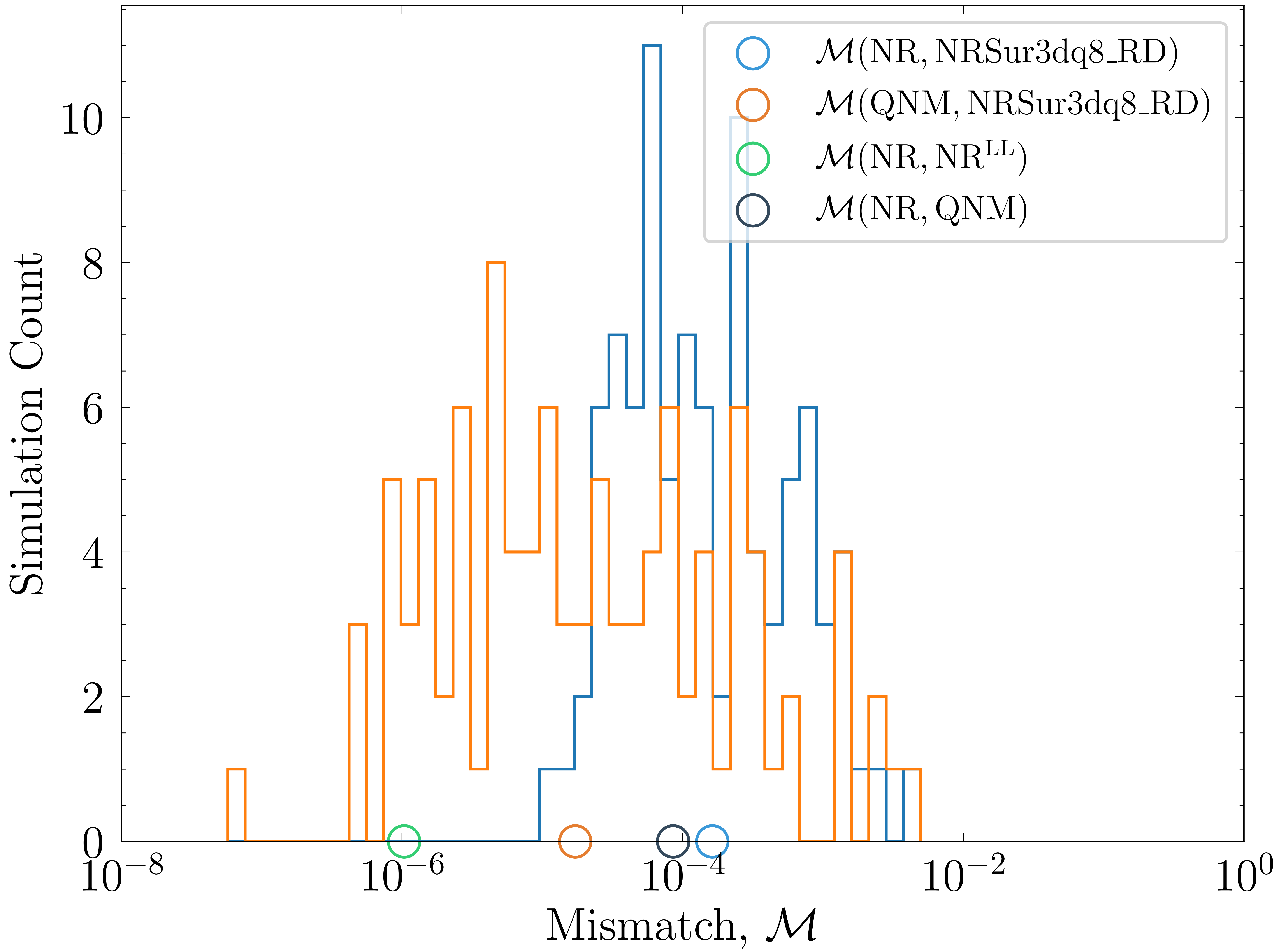
$$\epsilon(\Re[\mathscr{A}_{\ell mn}]) = \left| \frac{\Re[\mathscr{A}^{\mathrm{QNM}}_{\ell mn}] - \Re[\mathscr{A}^{\mathrm{surr}}_{\ell mn}]}{\Re[\mathscr{A}^{\mathrm{QNM}}_{\ell mn}]} \right|$$



Ringdown Waveform Reconstruction



Surrogate Mismatches



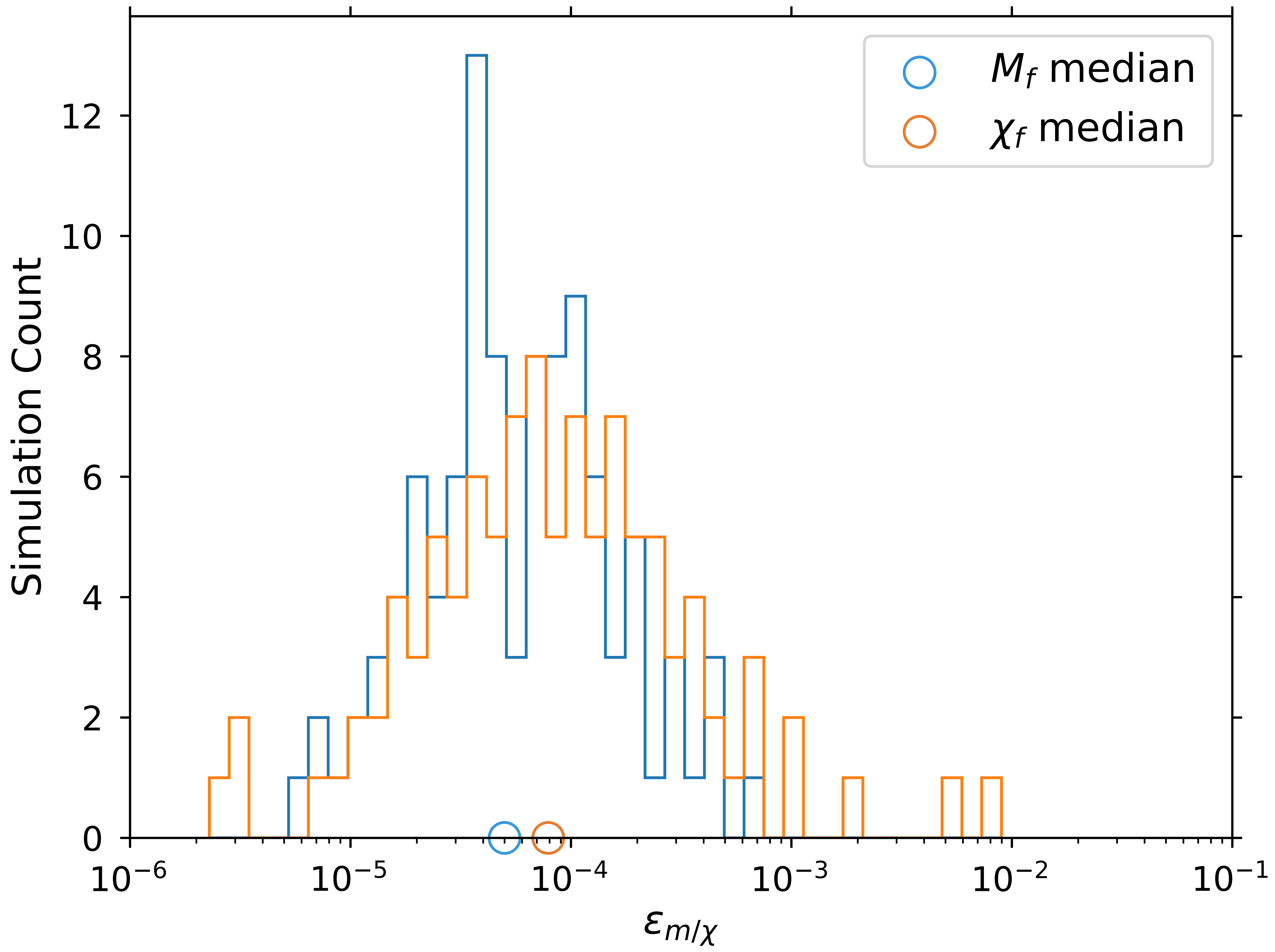
Parameter Recovery in NRSur3dq8_RD

- Errors in mass and spin are calculated as

$$\epsilon_m = \frac{|m_f^{\mathrm{NR}} - m_f^{\mathrm{surr}}|}{m_f^{\mathrm{NR}}}$$

$$\epsilon_\chi = |\chi_f^{\mathrm{NR}} - \chi_f^{\mathrm{surr}}|$$

- The results are on par with `NRHybSur3dq8`



Future Work: Waveform Models

- We are in the process of including quadratic modes to `qnmfits`.
- Ideally we would also include at least one of quadratic mode to `NRSur3dq8_RD` as well as retrograde modes.
- The holy grail would be a precessing surrogate without loss of accuracy, but that proves to be more complicated.

Summary

- Multimode ringdown models that take into account memory are necessary to take full advantage of future GW detections.
- Greedily modeling QNMs one can get up to 5 orders of magnitude improvement in model mismatches. Similar improvements can be seen by making CCE waveforms to the superrest frame.
- NRSur3dq8_RD can be incorporated into a full waveform surrogate for high-precision parameter estimation, making it a powerful tool that will prove crucial in next generation detectors.

Thank you for your attention!

`qnmfits` and `KerrRingdown` are packages for fitting QNM amplitudes. They have utilities for reading different waveform types including CCE waveforms from SXS.

`github.com/sxs-collaboration/qnmfits`
`github.com/cookgb/KerrRingdown`

qnmfits

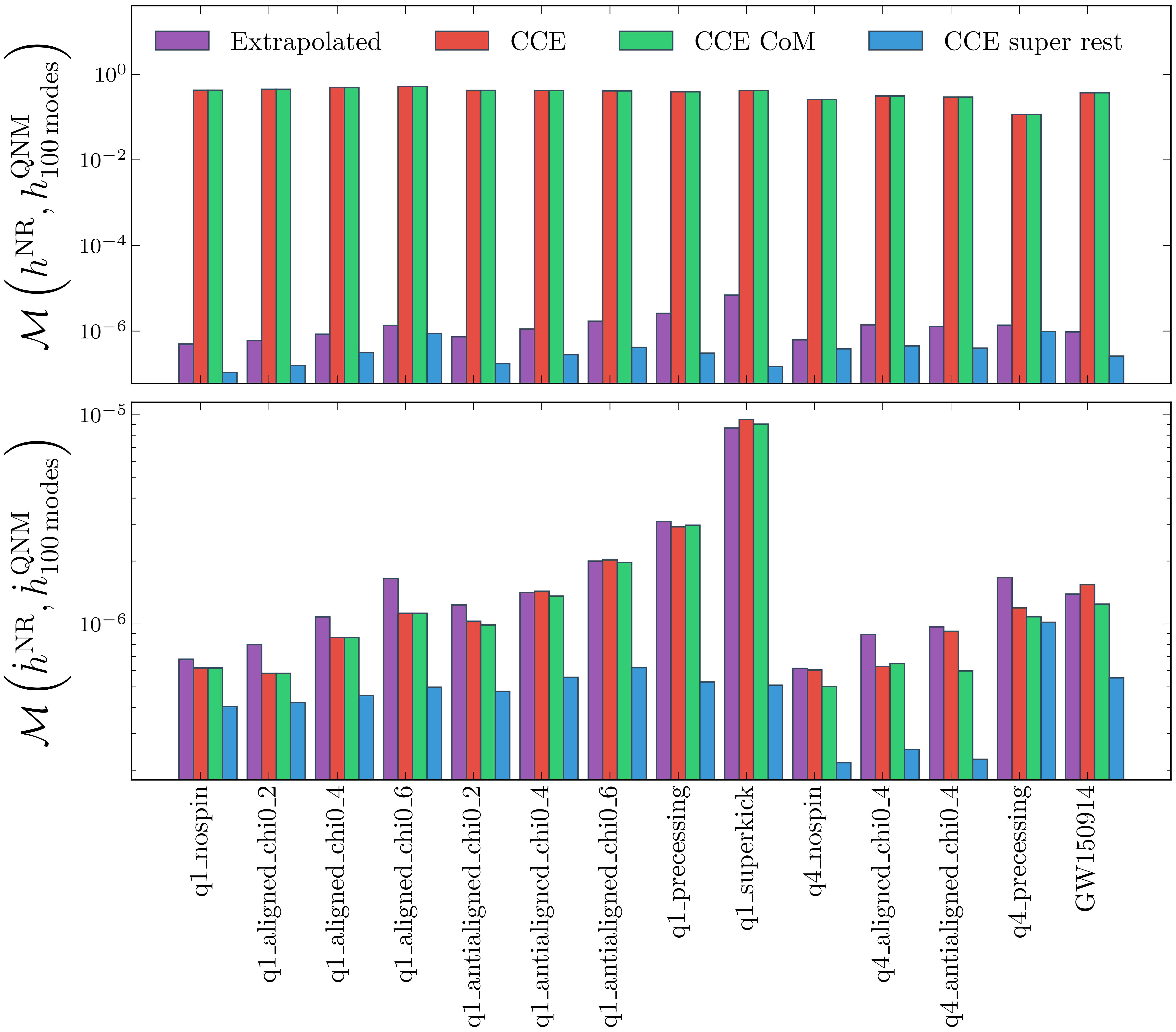
KerrRingdown

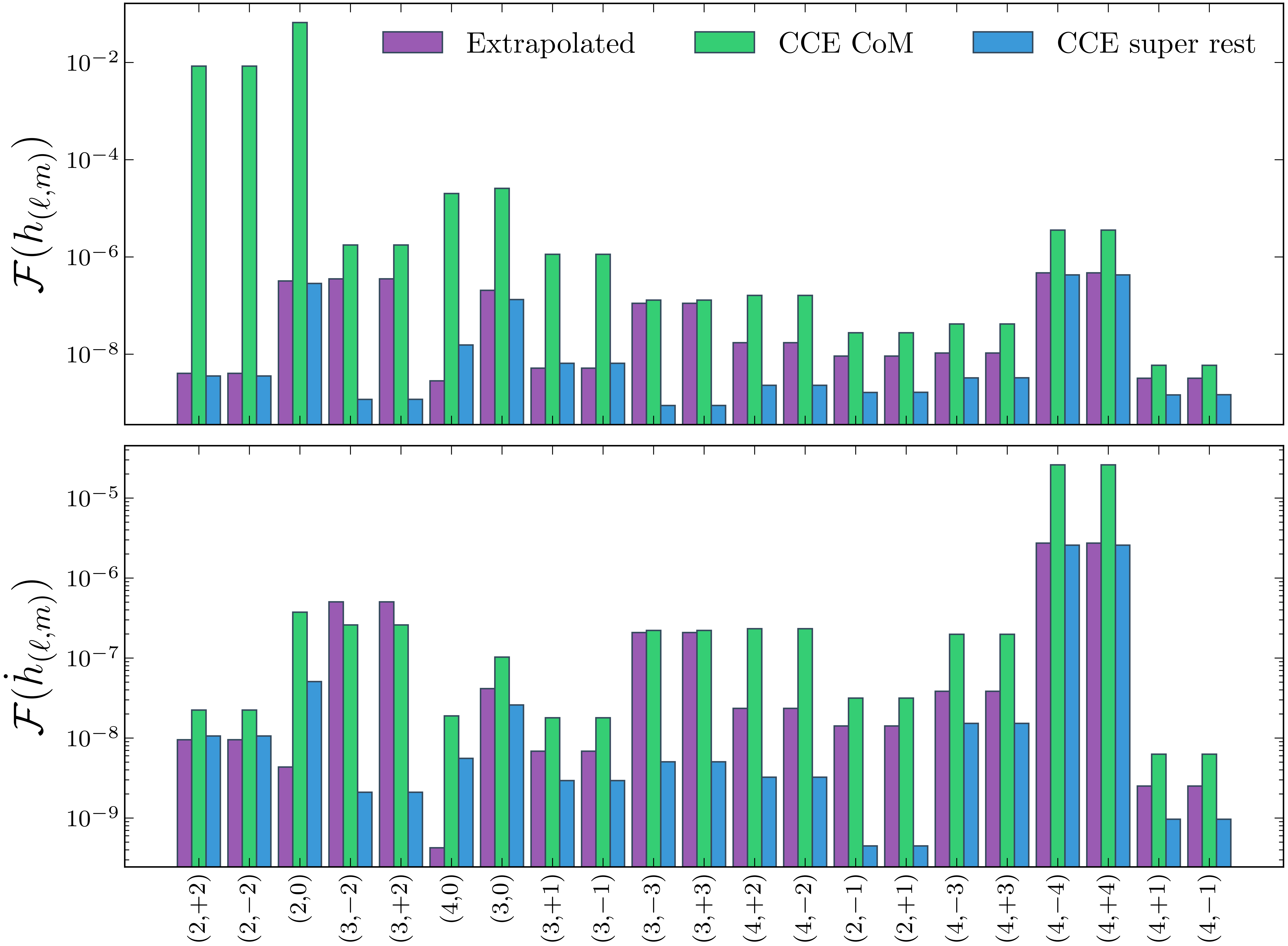
`NRSur3dq8_RD` allows you to predict QNM amplitudes and remnant mass and spin from initial parameters for spin-aligned systems.

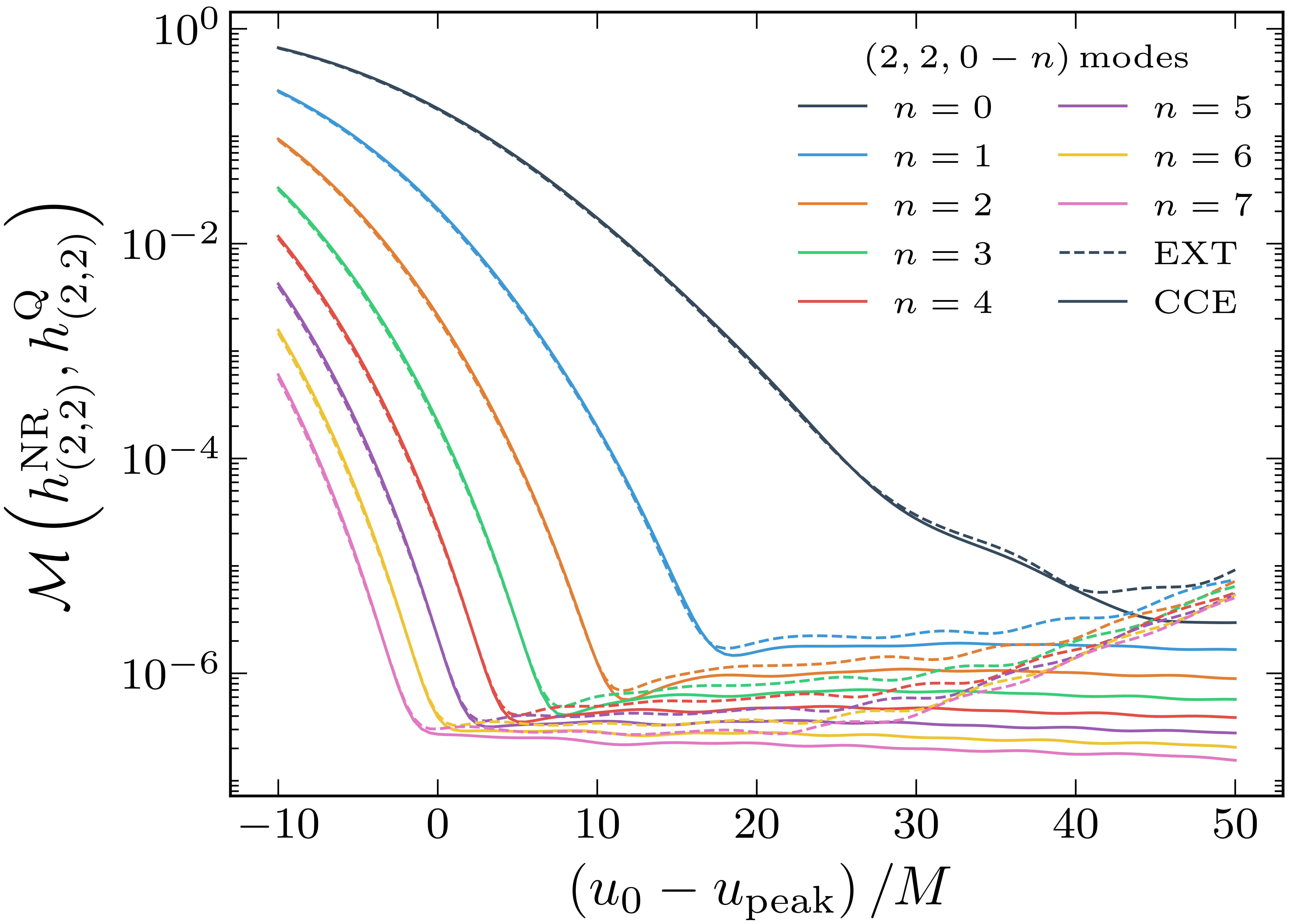
`github.com/vijayvarma392/surfinBH`

surfinBH

Extra Slides


Extrapolated
CCE
CCE CoM
CCE super rest
10^{0}
10^{-2}
10^{-4}
10^{-6}
$\mathcal{M}\left(h^{\rm NR}, h^{\rm QNM}_{100\,\rm modes}\right)$
10^{-5}
10^{-6}
$\mathcal{M}\left(\dot{h}^{\rm NR}, \dot{h}^{\rm QNM}_{100\,\rm modes}\right)$
q1_nospin
q1_aligned_chi0_2
q1_aligned_chi0_4
q1_aligned_chi0_6
q1_antialigned_chi0_2
q1_antialigned_chi0_4
q1_antialigned_chi0_6
q1_precessing
q1_superkick
q4_nospin
q4_aligned_chi0_4
q4_antialigned_chi0_4
q4_precessing
GW150914


Extrapolated
CCE CoM
CCE super rest
$\mathcal{F}(h_{(\ell,m)})$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
$\mathcal{F}(\dot{h}_{(\ell,m)})$
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
(2,+2) (2,−2) (2,0) (3,−2) (3,+2) (4,0) (3,0) (3,+1) (3,−1) (3,−3) (3,+3) (4,+2) (4,−2) (2,−1) (2,+1) (4,−3) (4,+3) (4,−4) (4,+4) (4,+1) (4,−1)


$(2, 2, 0-n)$ modes
$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$
$n = 6$
$n = 7$
EXT
CCE
10^0
10^{-2}
10^{-4}
10^{-6}
-10
0
10
20
30
40
50
$\mathcal{M}\left(h^{\rm NR}_{(2,2)}, h^{\rm Q}_{(2,2)}\right)$
$\left(u_0 - u_{\rm peak}\right)/M$