Building a theory of gravity around a given spacetime

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October 29, 2020, Tübingen

Introduction

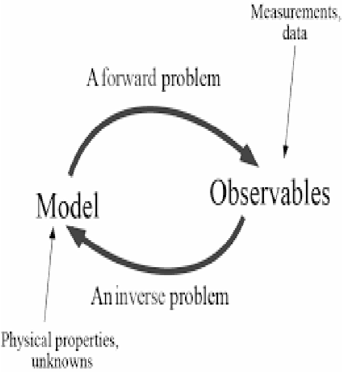
Black holes

Black holes are somehow paradoxically the simplest and most extreme objects within the universe. Although recent experiments (X-rays, GWs, QPOs...) find no evidence for a break-down of GR, there is a great deal of interest in finding and classifying black holes in modified theories of gravity (or even in GR with 'unusual' matter; no-hair breaks down for EYM [Bartnik & McKinnon, PRL **61**, 141 (1988)]).

Suppose that we did find evidence for a non-Kerr black hole...

How can this information be used to guide us towards the true theory of gravity? Same question applies for cosmological data (e.g., suppose Lambda-CDM failed to describe the universe).

Inverse problem



Starting from a particular field configuration, can one find an invariant Lagrangian density whose equations of motion admit that field as an exact solution? Owing to the complexity of the differential equations involved, which are typically non-linear in realistic problems, finding such a Lagrangian, let alone all Lagrangians, can be a challenging task, especially so in theories of gravity built from geometric invariants.

Example: cosmological reconstruction in $f(R)$ I

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$f(R)$ gravity – “non-linear” generalisation of general relativity

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \left[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right] f'(R) = \kappa^2 T_{\mu\nu}$$

– These theories are motivated from a variety of directions (Starobinsky inflation, ‘natural’ correction; why only linear?, derivable from string theory considerations)

– $f(R) = R$ returns general relativity; one could think of the function f as being the object one is trying to constrain.

Cosmological reconstruction in $f(R)$ gravity II

FLRW Cosmology

The first Friedmann equation in $f(R)$ gravity can be written

$$0 = -f(R)/2 + 3(H^2 + \dot{H})f'(R) - 18(4H^2\dot{H} + H\ddot{H})f''(R) + \kappa^2\rho, \quad (1)$$

where $H = \dot{a}/a$ is the Hubble parameter.

– For this simple case where the metric effectively only depends on time, the Ricci scalar $R = 6\dot{H} + 12H^2$ also only depends on time.

– The function f can thus be 'tuned' in any way you see fit to produce some particular cosmological evolution [S. Nojiri et al., Phys. Lett. B **681**, 74 (2009); E. Elizalde et al., Phys. Rev. D **77**, 106005 (2008)].

Bottom-up vs. Top-down approaches for BHs

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Top-down

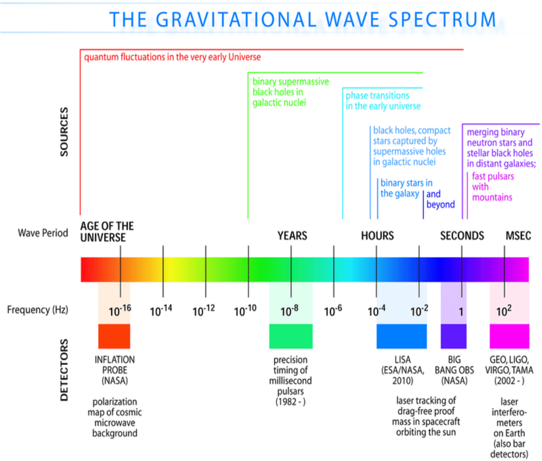
Pick a particular theory of gravity. Compare the solutions obtained within that theory with a suitable GR counterpart.

The nonlinear and (often) higher-than-second order nature of non-GR field equations renders the task of finding exact solutions challenging. To address this, various approaches to constructing metrics representing generic BHs in a theory-agnostic manner have been developed; hence, we have...

Bottom-up

Use a phenomenological parameterization of the spacetime that incorporates generic deformations of the GR counterpart. However, even if the deviation parameters of the parameterized metric can be constrained, bottom-up approaches do not necessarily guide one towards the true theory of gravity.

Bottom-up limitations



Bottom-up approaches cannot account for back-reaction effects in a totally self-consistent way; without an over-arching theory, equations for gravitational radiation cannot be derived (though see Völkel & Barausse PRD 2020). For electromagnetic-based tests (e.g., accretion disk physics and X-rays) this is less of a problem. [Cr: LIGO collaboration]

Static black holes

In the static case, at least for $f(R)$ gravity, the situation is rather similar to the simple cosmological case.

In Boyer-Lindquist coordinates (t, r, θ, ϕ), the line element may be written as

$$ds_g^2 = -A(r)^2 dt^2 + B(r)^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad (2)$$

where the vanishing of g^{rr} defines the location of the event horizon.

Inversion for static BHs in $f(R)$

We once again, like in the FLRW case, have only one coordinate which all equations depend on...

an inversion for f and R is straightforward. One can show that [Cappoziello et al. CQG **24**, 8 (2007)]

$$B(r) = \frac{r^2 f''(R)A'(r)R'(r) + 2rf'(R)A'(r) + 4rf''(R)A(r)R'(r) + 2Af'(R)}{A(r)\left[(2 + r^2 R(r))f'(R) - r^2 f(R)\right]}$$

Essentially, there is a total degeneracy between the function f and one of the metric coefficients.

Stationary spacetimes?

This trick will no longer work when there are less than 3 ignorable coordinates.

A general observation

If, for some particular R, we have that $f(R) = f'(R) = 0$, then the metric will be a vacuum solution to the $f(R)$ theories of gravity.

This constraint can only hold if R is constant, and then allows for a huge range of functions f, e.g.

$$f(R) = \sum_{i \geq 2} a_i (R - R_0)^i$$

if analyticity is demanded.

Yamabe problem

The previous 'trick' works to build a theory around a metric if the scalar curvature is constant. If, however, we take a conformal (frame) transformation of a general metric, $g \mapsto e^{2\varphi} g$, then the conformal scalar curvature reads

$$\tilde{R} = e^{-2\varphi} \left(R - 6\Delta\varphi - 6|d\varphi|^2 \right) . \tag{3}$$

Yamabe problem asks whether there exists a function φ such that $\tilde{R}$ is constant [known to be true in a number of cases, e.g. Lassas et al. Comm. Math. Phys. **360**, 555 (2018)]. In this case, the conformal metric is a solution to a huge family of $f(R)$ theories of gravity. This implies, amongst other things, that practically any conceivable null cone structure can emerge in *some* $f(R)$ theory.

Extending these ideas

With the previous concepts in mind, we can consider a new(?) theory of gravity, governed by the action

$$\mathcal{A} = \kappa \int d^4x\sqrt{-g}f\big(F(\phi)R + V(\phi) - \omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi\big), \qquad (4)$$

where $\kappa = (16\pi G)^{-1}$, G is Newton's (bare) constant, and F, V, and ω are potential functions of the scalar field ϕ.

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- $f(X) = X$: scalar-tensor theory
- $V = \omega = 0, F = 1$: pure $f(R)$ gravity
- Curvature-coupled modification of k-essence theory, where $\mathcal{A}_k \sim R + f(V(\phi) - \omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi)$.

Aspects of the theory I

EOM are qualitatively similar to those of $f(R)$ gravity. Variation with respect to the metric yields

$$\begin{aligned} 0 =& F(\phi)f'(X)R_{\mu\nu} - \frac{f(X)}{2}g_{\mu\nu} + g_{\mu\nu}\Box\left[F(\phi)f'(X)\right] \\ & - \nabla_\mu\nabla_\nu\left[F(\phi)f'(X)\right] - \omega(\phi)f'(X)\nabla_\mu\phi\nabla_\nu\phi, \end{aligned} \tag{6}$$

while variation with respect to ϕ gives

$$\begin{aligned} 0 =& f'(X)\Big[2\omega(\phi)\Box\phi + \frac{d\omega(\phi)}{d\phi}\nabla_\alpha\phi\nabla^\alpha\phi \\ & + R\frac{dF(\phi)}{d\phi} + \frac{dV(\phi)}{d\phi}\Big] + 2\omega(\phi)\nabla^\alpha\phi\nabla_\alpha f'(X), \end{aligned} \tag{7}$$

in vacuo.

Aspects of the theory II

In general, several conditions are imposed on scalar-tensor dynamics to ensure a well-defined theory. In the case of linear f, demanding that the graviton carries a positive energy and that the scalar field carries a non-negative kinetic energy requires that [Doneva et al., Phys. Rev. D **88**, 084060 (2013)]

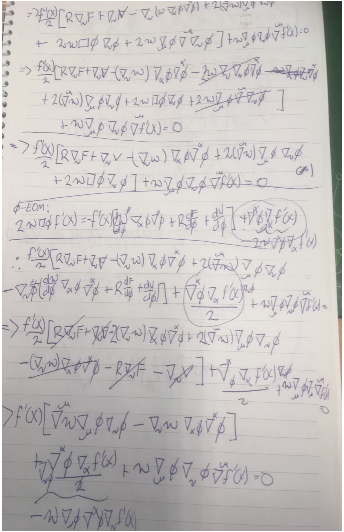
$$F(\phi) > 0$$

and

$$2F(\phi)\omega(\phi) + 3\left[dF(\phi)/d\phi\right]^2 \geq 0,$$

respectively. In the case of Brans-Dicke theory, which consists of the choices $F(\phi) = \phi$ and $\omega(\phi) \propto \phi^{-1}$, the aforementioned conditions are satisfied automatically for linear f.

Aspects of the theory III



Employing the Bianchi identities $\nabla_\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0$ and $\left(\Box \nabla_\nu - \nabla_\nu \Box \right) Z = R_{\mu\nu} \nabla^\mu Z$, the first of which is familiar from GR while the second is valid for any function Z, some extensive though not particularly difficult algebra shows that applying a contravariant divergence to the field equations produces terms which vanish identically. As such, for the non-vacuum case where a stress-energy tensor $T_{\mu\nu}$ occupies the left-hand side,, geometric identities give $\nabla^\mu T_{\mu\nu} = 0$ exactly, as in the pure $f(R)$ and scalar-tensor cases.

Main result

For *any given metric* g, if

i) a scalar field ϕ can be chosen such that $X = X_0$ for some constant X_0, and

ii) the function f satisfies $f(X_0) = f'(X_0) = 0$,

then g is a solution to the field equations. Note that these conditions are sufficient but not necessary; the Kerr metric, for example, can be an exact solution even when condition ii) is not satisfied [Psaltis et al., Phys. Rev. Lett. **100**, 091101 (2008)].

$$\begin{aligned} 0 =& F(\phi) f'(X) R_{\mu\nu} - \frac{f(X)}{2} g_{\mu\nu} + g_{\mu\nu} \Box \left[F(\phi) f'(X) \right] \\ & - \nabla_\mu \nabla_\nu \left[F(\phi) f'(X) \right] - \omega(\phi) f'(X) \nabla_\mu \phi \nabla_\nu \phi, \end{aligned} \tag{8}$$

A deformed Kerr spacetime

Suppose that astrophysical data implied that black holes were described by a simple generalization of the Kerr metric whose line element, in Boyer-Lindquist coordinates (t, r, θ, φ), reads

$$\begin{aligned} ds^2 =& \frac{a^2 \sin^2\theta - \Delta}{\Sigma} dt^2 - \frac{2a \sin^2\theta \left(a^2 + r^2 - \Delta\right)}{\Sigma} dt d\varphi \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\left(a^2 + r^2\right)^2 - a^2 \sin^2\theta \Delta}{\csc^2\theta \Sigma} d\varphi^2, \end{aligned} \tag{9}$$

where $\Delta = r^2 - 2Mr + a^2 + \epsilon M^3/r$ and $\Sigma = r^2 + a^2 \cos^2\theta$. In expression (9), M and a denote the mass and spin of the black hole, respectively, while ϵ is a dimensionless 'hair'. The metric (9) admits an (outer) event horizon at the largest positive root of $\Delta = 0$, which occurs near the Kerr value for sufficiently small ϵ, viz. $r \approx M + \sqrt{M^2 - a^2} + \mathcal{O}(\epsilon)$.

Mixed scalar-$f(R)$ theory inverse

The mixed scalar-$f(R)$ theory with $f(X) = X^{1+\delta}$ for any $\delta > 0$ admits the deformed-Kerr metric as an exact solution, provided that the scalar field ϕ solves the kinematic constraint equation

$$0 = F(\phi)R + V(\phi) - \omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi. \tag{10}$$

Note that this function f is not analytic, though $f = 0$ is a critical point at $X = 0$ for $\delta > 0$ [compare with the analytic function(s) written down before.] The deformation parameter ϵ may thus be thought of as a kind of 'scalar hair'.

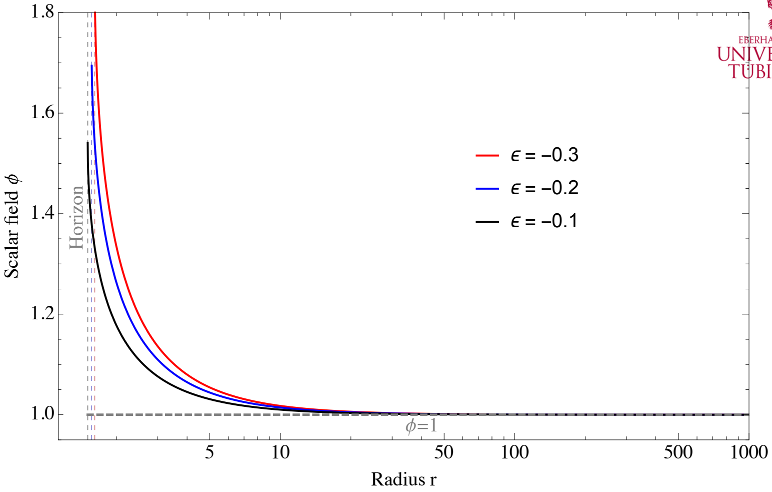


Figure: A sample of radial scalar-field profiles as solutions to (10) for $M = 1$ and $a = 0.9$, for the Brans-Dicke choices.

This seems to suggest that...

- There is a well-behaved scalar field profile for all values of the deformation parameter for which the metric is an exact solution to the theory.

and....

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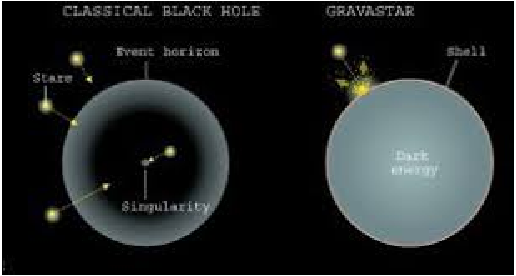
- The theory has such a large configuration space, that virtually all metrics can be included for some choices of the free functions.

with the added benefit that...

- There is a well-behaved scalar field profile for all values of the deformation parameter for which the metric is an exact solution to the theory.
- The theory has such a large configuration space, that virtually all metrics can be included for some choices of the free functions.
- Astrophysical disturbances of a generic spacetime can be studied self-consistently (though this is non-trivial).

Philosophical curiosities I

- Notice we only talked about constructing vacuum solutions. Since the seed metric could arise as a matter-filled solution in GR (for example), this implies that vacuum gravitational fields in this theory can imitate the gravitational fields of material bodies in a different theory. [Cr: APlusPhysics.com]



Philosophical curiosities II

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- Notice we only talked about constructing vacuum solutions. Since the seed metric could arise as a matter-filled solution in GR (for example), this implies that vacuum gravitational fields in this theory can imitate the gravitational fields of material bodies in a different theory.
- Suppose we studied a perturbation; by the same arguments as presented before, this perturbed geometry could also arise as an exact solution in some theory. In this way, a fully dynamical universe filled with stars and GWs might even be thought of as an exact solution (degeneracy problem?).

Some further issues to consider

It is important to note that we do have not commented here on the physical viability or otherwise of such theories.

Further analysis is required to determine whether there exists members of the class constructed here that can accommodate existing (and upcoming) astrophysical experiments. For example, there may be no such f which simultaneously satisfies the inverse criterion and passes Solar System and/or strong-field tests, even with screening mechanisms. However, the exact conservation of energy-momentum hints that this may be possible.

This is also far from a general solution to the inverse problem! Just one particular class of solutions.