Temperature of Horndeski Black Holes

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Based on my recent paper with
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Outline

- Review of BH thermodynamics
- Wald entropy and its ambiguities
- Solution Phase Space Method
- Some words on Horndeski theories
- BH temperature in Horndeski theories
- Summary and Outlook

Black holes.....

- constitute a generic class of solutions to general theories of gravity;

- are characterized by their horizon.

- Horizon is boundary of outgoing causal curves in the outside region and is a smooth, non-singular codimension one null hypersurface.

- BHs are generically localized sources of (conserved) charges like mass, angular momentum, electric and magnetic charges.

- Studying classical stationary BH solutions in GR, revealed that BH's obey four laws which closely resemble laws of thermodynamics. [Bardeen, Carter and Hawking (1973)]

- Studying thermodynamics in presence of BH horizons, led to the fact that there should be an entropy associated with BH to save 2nd law of thermodynamics. [Bekenstein (1972)]

- For BH solutions to GR, Bekenstein entropy is proportional to the horizon area.

- Studying quantum fields on a BH background while ignoring back-reaction effects, uncovered that BHs radiate a perfect black body radiation, like any thermodynamical system. [Hawking (1974-5)]

- For stationary BHs with a Killing horizon, the Hawking temperature is surface gravity κ divided by 2π.

- Combining Bekenstein's and Hawking's results for Killing horizons in GR ($\hbar = c = 1$):

$$S_{\mathsf{BH}} = \frac{A_{\mathsf{H}}}{4G_{\mathsf{N}}}, \qquad T_{\mathsf{BH}} = \frac{\boldsymbol{\kappa}}{2\pi} \tag{1}$$

- Hawking temperature can be locally associated with any accelerated observer, as implied by Einstein's Equivalence Principle (EEP). [Unruh (1976)]

- For BHs with bifurcate Killing horizons,

  – BH entropy is a conserved charge. [Wald (1993)]

  – a general formula for entropy of BH solutions to *any diff. invariant theory*, assuming BH temperature is $\frac{\boldsymbol{\kappa}}{2\pi}$. [Wald (1993)]

  – first law of thermodynamics for BHs in generic diff inv. gravity theories. [Iyer-Wald (1994)]

- If we denote BH mass, angular momentum and electric charge by M, J and Q, then:

$$\delta M = T_{\text{BH}}\ \delta S_{\text{BH}} + \Omega_{\text{H}}\ \delta J + \Phi_{\text{H}}\ \delta Q \tag{2}$$

$\Omega_{\text{H}}, \Phi_{\text{H}}$ are horizon angular velocity and horizon electric potential.

- "Chemical potentials" $T_{\text{BH}}, \Omega_{\text{H}}, \Phi_{\text{H}}$ are
  - properties attributed to the horizon and subject to the first law of BH thermodynamics, and
  - are geometric quantities, only depend on the solution and not the theory, but are observer dependent.

- Charges depend on the solution and the theory.

- Among all the charges, entropy is different, as it is dimensionless and observer independent.

Wald Entropy.....

- works for bifurcate Killing horizons,
- is the Noether-Wald conserved charge associated with the Killing vector field which generates the horizon ζ_H,
- depends only on the BH solution (metric) and the gravitational part of the action,
- *it uses surface gravity=temperature as an input/assumption*,
- for Einstein GR it reduces to the Bekenstein-Hawking area law.

Short review of derivation of the Wald Entropy

- Consider
  - a generally covariant gravity theory governed by the Lagrangian (top-form) $\mathbf{L}(g_{\mu\nu}, \nabla_\mu; \Phi)$, and
  - a BH solution to this theory with bifurcate Killing horizon generated by ζ_{H}.

- Surface gravity $\boldsymbol{\kappa}$ is then defined as

$$(\mathrm{d}\zeta_{\mathsf{H}})_{\mu\nu} := 2\boldsymbol{\kappa}\ \boldsymbol{\epsilon}_{\mu\nu} \tag{3}$$

$\boldsymbol{\epsilon}_{\mu\nu}$ is the bi-normal two form to the codimension 2 bifurcation surface of the horizon $\mathcal{H}$, normalized as

$$\boldsymbol{\epsilon}^{\mu\nu}\ \boldsymbol{\epsilon}_{\mu\nu} = -2$$

- Noether-Wald conserved charge associated with ζ_{H} is

$$S_{\mathsf{BH}}^{\mathsf{W}} = Q_{\zeta_{\mathsf{H}}} = 2\pi \oint_{\mathcal{H}} \mathbf{Q}_{\mathsf{Wald}} \tag{4}$$

where $\mathbf{Q}_{\mathsf{Wald}}$ is the (d-2)-form:

$$\mathbf{Q}_{\alpha_1\alpha_2\cdots\alpha_{d-2}} = \mathbf{Q}^{\rho\lambda}\ \epsilon_{\rho\lambda\alpha_1\alpha_2\cdots\alpha_{d-2}}$$

$$\mathbf{Q}^{\rho\lambda} := -\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\lambda}}\ \boldsymbol{\epsilon}_{\mu\nu} = -\frac{1}{2\boldsymbol{\kappa}}\ \frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\lambda}}\ (\mathsf{d}\zeta_{\mathsf{H}})_{\mu\nu}$$

- Noether-Wald charge density has various ambiguities:

$$\mathbf{Q} = \mathbf{Q}_{\mathsf{Wald}} + \boldsymbol{W}(\Phi, \partial\Phi) \cdot \zeta_{\mathsf{H}} + \mathsf{d}\boldsymbol{Z}(\Phi, \partial\Phi; \zeta_{\mathsf{H}}) + \boldsymbol{Y}(\Phi, \delta_{\zeta_{\mathsf{H}}}\Phi),$$

$\boldsymbol{W}, \boldsymbol{Z}$ terms are linear in ζ_{H} and $\boldsymbol{Y}$ is linear in $\delta_{\zeta_{\mathsf{H}}}\Phi$.

- $\boldsymbol{W}$ ambiguity comes from shift of Lagrangian by a total derivative: $\mathbf{L} \to \mathbf{L} + \mathsf{d}\boldsymbol{W}$.

- One needs to argue that these ambiguities do not contribute.

- Recall that

  - $\delta_{\zeta_\mathsf{H}}\Phi = 0$ everywhere since ζ_H is a Killing (global symmetry of the solution), and

  - At $\mathcal{H}$, $\zeta_\mathsf{H} = 0$.

- So, if any $\boldsymbol{W}, \boldsymbol{Z}, \boldsymbol{Y}$ constructed form the background field solutions computed at the horizon $\mathcal{H}$ are finite, the ambiguities cancel out.

- Wald argued that this condition holds as the BH solution is smooth at $\mathcal{H}$ and hence no such ambiguities.

- Nonetheless, smoothness of horizon, as we show explicitly, does not necessarily imply vanishing of these ambiguities.

- $\boldsymbol{W} \cdot \zeta_{\mathsf{H}}$ can remain finite for certain BH solutions in certain theories, despite having smooth horizons.

- We show this happens for a class of BH solutions to Horndeski theories, where we explicitly construct such a $\boldsymbol{W}$ ambiguity term.

- Non-vanishing ambiguity means Wald's entropy formula and hence Wald's derivation of the first law, should be revisited in these cases.

- To resolve the issue, we use Solution Phase Space Method (SPSM) for computing conserved charges, which again Wald had an important role in its development.

- SPSM does not suffer from the $\boldsymbol{W}$ ambiguity issue, but gives charge variations and to have well-defined charges we need to impose the integrability condition.

- Our main result is that

integrability condition for the entropy requires revising
surface gravity= temperature
assumption.

Brief review on Solution Phase Space Method (SPSM)

- SPSM is based on covariant phase space method (CPSM) which was developed by Lee & Wald (1991) and then expanded by many others in particular Wald (1993); Iyer & Wald (1994); Barnich & Brandt (2002) and Barnich & Compere (2008).

- SPSM and its power to analyze BH thermodynamics was discussed in Hajian & MMShJ (2015).

- CPSM asserts that

  All field configurations (histories) may form a Phase Space, with a symplectic structure systematically constructed from the action of the theory.

- Consider a field configuration Φ and perturbations around it $\delta\Phi$.

- On-shell field configurations $\bar{\Phi}$ satisfy field equations and on-shell perturbations $\delta\Phi$ satisfy linearized field equations.

- Set of $\bar{\Phi}$ may be viewed as coordinates on a phase space and $\delta\Phi$ as one-forms in the corresponding cotangent space.

- On-shell cotangent space includes two important directions:
  - $\delta\Phi$ generated by gauge and/or diffeo's transformations on $\bar{\Phi}$;
  - parametric variations, generated by moving in the parameter space of the solutions $\bar{\Phi}$, e.g. the difference between two Sch'd solutions with masses m and $m + \delta m$.

▶ Symplectic structure

- Let $\boldsymbol{\omega}$ be a *finite*, *on-shell closed*, *nondegenerate* $(d-1;2)$-form, a $d-1$ form in spacetime and a two-form over tangent space :

$$\boldsymbol{\omega} = \boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \bar{\Phi}]$$

- Symplectic structure $\boldsymbol{\Omega}_{\Sigma}$ is a $(0;2)$-form, defined by integration of symplectic current $\boldsymbol{\omega}$ over a Cauchy surface Σ:

$$\boldsymbol{\Omega}_{\Sigma}\,[\delta_1\Phi, \delta_2\Phi; \bar{\Phi}] = \int_{\Sigma} \boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \bar{\Phi}]$$

- Presymplectic potential $\boldsymbol{\theta}[\delta\Phi; \Phi]$: $\boldsymbol{\omega} = \delta\boldsymbol{\theta}$, or

$$\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi] = \delta_1\boldsymbol{\theta}[\delta_2\Phi; \Phi] - \delta_2\boldsymbol{\theta}[\delta_1\Phi; \Phi]$$

$\boldsymbol{\theta}$ is a $(d-1;1)$-form.

- The Lee-Wald contribution to $\boldsymbol{\theta}$:

$$\delta \boldsymbol{L}\Big|_{on-shell} = \mathsf{d}\boldsymbol{\theta}_{(LW)}.$$

- Ambiguity of adding total derivatives to Lagrangian, $\boldsymbol{L} \to \boldsymbol{L} + \mathsf{d}\mathbf{W}$, leads to an ambiguity in $\boldsymbol{\theta}$,

$$\boldsymbol{\theta} \;\to\; \boldsymbol{\theta} + \delta\mathbf{W}.$$

- Moreover, $\boldsymbol{\theta}$ is ambiguous up to *boundary terms* $\mathbf{Y}$:

$$\boldsymbol{\theta} = \boldsymbol{\theta}_{(LW)} + \mathsf{d}\mathbf{Y}.$$

$\mathbf{Y} = \mathbf{Y}(\Phi, \delta\Phi)$ is a $(d-2;1)$-form.

- $\mathbf{W}$ does not change symplectic potential $\boldsymbol{\omega}$, while

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{(LW)} + \mathsf{d}\delta\mathbf{Y}.$$

- Therefore, symplectic form is free of $\mathbf{W}$ ambiguity while the $\mathbf{Y}$ still needs to be dealt with.

- There is a "$\mathbf{Z}$ ambiguity" which will appear when we discuss the surface charges.

- Consistency of symplectic structure requires

  - Conservation:

    $$\mathrm{d}\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi] \approx 0 \qquad \text{for all on-shell fields and perturbations.}$$

  - Non-degeneracy: $\mathbf{\Omega}_\Sigma$ has no degenerate directions, is conserved and is independent of Σ.

Surface charges

- Fundamental Theorem of Covariant Phase Space Method

$$\boldsymbol{\omega}\ [\delta\Phi, \delta_\chi\Phi; \Phi] \approx \mathrm{d}\boldsymbol{K}_\chi[\delta\Phi; \Phi]$$

where $\delta_\chi\Phi$ is a specific transformation generated by a symmetry χ, and $\boldsymbol{K}$ is a $(d-2;1)$-form.

- The above defines $\boldsymbol{K}$ up to a $\mathbf{Z}$ ambiguity, $\boldsymbol{K} \to \boldsymbol{K} + \mathrm{d}\mathbf{Z}$.

- Given $\boldsymbol{K}$ one can define charge variations:

$$\not{\delta} Q_\chi = \oint_{\partial\Sigma} \boldsymbol{K}_\chi[\delta\Phi; \bar{\Phi}]$$

where $\partial\Sigma$ is the boundary of Σ.

- If $\partial\Sigma$ is compact, $\mathbf{Z}$ ambiguity does not affect the charge variation.

- If $\not{\delta} Q_\chi$ is non-zero on the phase space, χ a physical symmetry generator.

- $\not{\delta} Q_\chi$ is a $(0;1)$-form.

- Charge Q_χ is integrable if $\not{\delta} Q_\chi$ is an exact form. The necessary condition for that is,

$$\delta_1 \not{\delta}_2 Q_\chi - \delta_2 \not{\delta}_1 Q_\chi = 0$$

- Integrability [Lee-Wald '1991]:

$$\oint_{\Sigma} \chi \cdot \boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \bar{\Phi}] = 0, \quad \forall \chi, \delta\Phi$$

There usually exist $\mathbf{Y}$ terms which guarantee the above.

- Using integrability one can define surface charges Q_χ:

$$Q_\chi[\Phi] = \int_\gamma \oint_{\partial\Sigma} \boldsymbol{K}_\chi[\delta\Phi; \Phi] + N_\chi[\Phi]$$

where N is the zero point charge.

- Q_χ is then the Hamiltonian associated with transformations generated by χ.

▶ For BH thermodynamics we restrict ourselves to

– "exact symmetries" (Killing vectors) for which $\delta_\xi \Phi = 0$ and,

– the cotangent space to be parametric variations.

- In this case the charge variation is symplectic and $\partial\Sigma$ can be any compact, spacelike codimension 2 surface in spacetime.

- $\partial\Sigma$ need not necessarily be $d-2$ dimensional (celestial) sphere at infinity, i^0 or $\mathcal{H}$; these charges are symplectic and may be defined at any radius [Hajian & MMShJ (2015)].

- For the BH entropy and ζ_{H} we may choose $\partial\Sigma = \mathcal{H}$ to remove the $\mathbf{Y}$ ambiguity.

- But, we still need to check integrability of the charge.

Review of Horndeski theories

- Horndeski gravity is a family of scalar-tensor gravity theories with second order field equations and in general governed by the action:

$$S_{\text{Horn.}} = \frac{1}{16\pi G_{\text{N}}} \int \mathrm{d}^d x \sqrt{-g}\, \mathcal{L}_{\text{Horn.}}$$

$$\begin{aligned}\mathcal{L}_{\text{Horn.}} = {} & \mathcal{G}_2(\phi,\mathcal{X}) - \mathcal{G}_3(\phi,\mathcal{X})\Box\phi + \mathcal{G}_4(\phi,\mathcal{X})R + \mathcal{G}_4'(\phi,\mathcal{X})\left((\Box\phi)^2 - (\partial_{\mu\nu}\phi)^2\right) \\ & - \mathcal{G}_5(\phi,\mathcal{X})G^{\mu\nu}\partial_{\mu\nu}\phi - \frac{\mathcal{G}_5'(\phi,\mathcal{X})}{6}\left((\Box\phi)^3 + 2(\partial_{\mu\nu}\phi)^3 - 3\Box\phi(\partial_{\mu\nu}\phi)^2\right)\end{aligned}$$

  where $G_{\mu\nu}$ is the Einstein tensor, $\partial_{\mu\nu}\phi = \nabla_\mu\nabla_\nu\phi$, $\Box\phi = g^{\mu\nu}\partial_{\mu\nu}\phi$, $\mathcal{X} := -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ and $\mathcal{G}_i' = d\mathcal{G}_i/d\mathcal{X}$.

- In our conventions $\mathcal{G}_4(\phi = 0, \mathcal{X} = 0) = 1$. This is how we define the Newton constant G_{N}.

- Let's consider a family of Horndeski theories:

$$\mathcal{L}_{\text{Horn.}} = \mathcal{G}_2 + \mathcal{G}_3 + (\mathcal{G} - \mathcal{G}'\mathcal{X})R + \mathcal{G}' G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

- This theory has been studied a lot, in the context of cosmology as it may provide a good model for dark energy.

- In this theory speed of gravitons is different than speed of light.

- Photon to graviton speed ratio was severely constrained by the LIGO/Virgo GW170817 two neutron stars merger into BH,

$$|\frac{c_g}{c_\gamma} - 1| \lesssim 10^{-15}$$

removing the prospect of being a viable dark energy model.

▶ W ambiguity in Horndeski theory

- Recall Horndeski theories:

$$\mathcal{L}_{\text{Horn.}} = \mathcal{G}_2 + \mathcal{G}_3 + (\mathcal{G} - \mathcal{G}'\mathcal{X})R + \mathcal{G}' G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

- Since $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$, we have terms like $R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$:

$$R^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi = -(\nabla_\rho\nabla_\sigma\phi)^2 + (\Box\phi)^2 + \nabla_\mu\big(\nabla_\nu\phi\nabla^\nu\nabla^\mu\phi - \Box\phi\nabla^\mu\phi\big).$$

- Explicit dependence on the Ricci tensor $R_{\mu\nu}$ can be removed in favor of terms with ϕ derivatives and hence $\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\alpha\beta}}$ which appears in the Wald entropy. This leads to ambiguous due to a $\mathbf{W}$ ambiguity:

$$(\mathbf{Q}^{\mu\nu})_{\mu_3\ldots\mu_n} \to (\mathbf{Q}^{\mu\nu})_{\mu_3\ldots\mu_n} + \lambda\frac{\delta R^{\rho\sigma}}{\delta R_{\alpha\beta\mu\nu}}\partial_\rho\phi\partial_\sigma\phi\ \boldsymbol{\epsilon}_{\alpha\beta\mu_3\ldots\mu_n}.$$

- To circumvent this problem in Horndeski gravity, we use the solution phase space method.

Entropy of Horndeski black holes, SPSM

- Variation of the Hamiltonian generator associated with flows of ζ_{H}:

$$\delta\!\!\!/ H_{\zeta_{\text{H}}} = \int_{\mathcal{H}} \boldsymbol{k}_{\zeta_{\text{H}}}(\delta\Phi, \bar{\Phi}).$$

- The entropy variation δS_{BH} which satisfies a consistent first law is

$$\delta\!\!\!/ H_{\zeta_{\text{H}}} := T_{\text{BH}}\delta S_{\text{BH}} \quad \Longrightarrow \quad \delta S_{\text{BH}} := \frac{1}{T_{\text{BH}}}\int_{\mathcal{H}} \boldsymbol{k}_{\zeta_{\text{H}}}(\delta\Phi, \bar{\Phi}),$$

- T_{BH} is the black hole temperature and should be a purely geometric quantity and a constant over $\mathcal{H}$.

- Above definition yields an entropy variation free of $\mathbf{W}$ ambiguity.

- Key question is to find T_{BH} which makes the entropy integrable.

- The usual $T_{\text{BH}} = \frac{\kappa}{2\pi}$, does not work for Horndeski theory, as $c_g \neq 1$.

Speed of graivtons in Horndeski theories

- While generally covariant, Horndeski theories do not obey Strong Equivalence Principle.

- Gravitons do not necessarily move on the null geodesics, while photons can be moving on null geodesics.

- Speed of gravitons c_g depends on the background and to find it one should linearize the theory around a given background solution.

- Here we present a shortcut for computation of c_g.

► "$\phi + 3$" decomposition of Horndeski theories

- Consider a field configuration $g_{\mu\nu}, \phi$ and let

$$g_{\mu\nu} = h_{\mu\nu} + \sigma\phi_\mu\phi_\nu, \qquad \phi_\mu := \frac{\partial_\mu\phi}{|\partial\phi|}$$

σ is sign of $\phi_\mu\phi^\mu$, $\sigma = -1$ for cosmological backgrounds and $+1$ for black holes, and
$h_{\mu\nu}$ is the metric along constant ϕ surface, $h_{\mu\nu}\phi^\nu = 0$.

- "$\phi + 3$" decomposed Lagrangian is [arXiv:1304.4840]

$$\mathcal{L} = \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}\ {}^{(3)}\!R + (\mathcal{G} - 2\mathcal{X}\mathcal{G}')(K_{\mu\nu}K^{\mu\nu} - K^2)$$
$$+ 2\sqrt{-2\mathcal{X}}\ \mathcal{G}_{,\phi}\ K + \text{total derivative terms}$$

${}^{(3)}\!R$ is the scalar curvature of $h_{\mu\nu}$, $\mathcal{G}_{,\phi} = d\mathcal{G}/d\phi$ and
K is the extrinsic curvature of our constant ϕ surfaces,

$$K_{\mu\nu} = h^\alpha_\mu\ \nabla_\alpha\phi_\nu,\ \ K = K^\mu{}_\mu$$

- For a black hole
  - ϕ_μ is typically along the "radial direction" and is normal to the horizon,
  - time direction is in the "3" part, along $h_{\mu\nu}$ and normal to ϕ_μ.
  - One can directly read the speed of gravitons:

$$c_g^2 = \begin{cases} \frac{\mathcal{G}-2\mathcal{X}\mathcal{G}'}{\mathcal{G}} & \text{for gravitons moving along } \phi_\mu \\ \\ \frac{\mathcal{G}}{\mathcal{G}} = 1 & \text{for gravitons moving normal to } \phi_\mu \end{cases}$$

- For cosmological FLRW backgrounds, ϕ_μ is timelike and

$$c_g^2 = \frac{\mathcal{G}}{\mathcal{G}-2\mathcal{X}\mathcal{G}'} \text{ for all gravitons.}$$

Effective Metric for Gravitons (EMG)

- Given direction dependent speed of gravitons, one may ask what is the "effective metric" $\mathrm{g}_{\mu\nu}$ where the gravitons move on its null rays, $\mathrm{g}^{\mu\nu} k_\mu k_\nu = 0$.

- It is easy to write this metric:

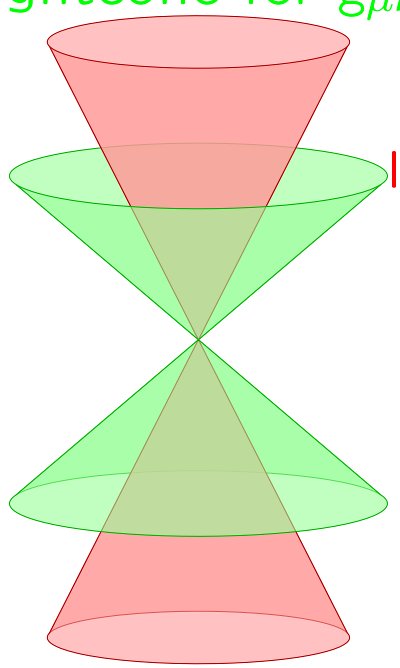
$$\mathrm{g}_{\mu\nu} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')g_{\mu\nu} - \mathcal{G}'\partial_\mu\phi\partial_\nu\phi$$

$$= (\mathcal{G} - 2\mathcal{X}\mathcal{G}')h_{\mu\nu} + \mathcal{G}\phi_\mu\phi_\nu$$

- To avoid a singular effective metric we assume $\mathcal{G}', \mathcal{G} - 2\mathcal{X}\mathcal{G}' \neq 0$.

- The above is a disformal map from original spacetime metric.

■ Depiction of lightcones for gravitons and other fields



For our examples $c_g^2 \leq 1$

Well-posedness of Horndeski dynamics requires the above form of light-cones, Kovacs & Reall, arXiv:2003.08389; Reall, arXiv:2101.11623

■ Effective surface gravity for gravitons

- Surface gravity as seen by gravitons for a horizon generated by Killing vector ζ^μ_{H} is

$$d\zeta_{\mathsf{H}} = 2\kappa\, c_g\, \mathcal{E} \qquad \text{at the horizon,}$$

- where,
  - ζ_{H} is normal to ϕ_μ at the horizon,
  - κ is the surface gravity in the matter metric $g_{\mu\nu}$,
  - $\mathcal{E}$ is the bi-normal tensor to the bifurcation surface $\mathcal{H}$, normalized as $\mathcal{E}_{\mu\nu}\mathcal{E}^{\mu\nu} = -2$.
  - Indices on $\zeta^\mu_{\mathsf{H}}, \mathcal{E}$ are lowered and raised by the effective graviton metric $\mathrm{g}_{\mu\nu}$.

- We need to rewrite the above in terms of $\boldsymbol{\epsilon}$, the volume two form of the original metric $g_{\mu\nu}$.

- Recalling the disformal map,

$$\mathcal{E} = \sqrt{\mathcal{G}(\mathcal{G} - 2\mathcal{X}\mathcal{G}')}\ \boldsymbol{\epsilon}.$$

- Inserting this into the above we obtain

$$\mathrm{d}\zeta_{\mathrm{H}} = 2\boldsymbol{\kappa}\ (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\ \boldsymbol{\epsilon},$$

- Effective graviton surface gravity $\boldsymbol{\kappa}_g$ is hence

$$\boldsymbol{\kappa}_g = \boldsymbol{\kappa}\ (\mathcal{G} - 2\mathcal{X}\mathcal{G}').$$

- The above suggests

$$T_{\mathrm{gravitons}} := \frac{\boldsymbol{\kappa}_g}{2\pi} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\ T_0, \qquad T_0 = \frac{\boldsymbol{\kappa}}{2\pi} \tag{5}$$

Example 1: Charged BH in Horndeski theory

- Consider the Horndeski-Maxwell theory,

$$\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}}\Big(R - F_{\mu\nu}F^{\mu\nu} + 2\gamma\ G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big)$$

- This action corresponds to $\mathcal{G}_2 = 0, \mathcal{G}_4 = 1, \mathcal{G}_5 = 2\gamma\phi$, which yields $\mathcal{G} = 1 + 2\gamma\mathcal{X}$ and $F = dA$ is the electromagnetic field strength.

- This theory has a spherically symmetric charged black hole solution:

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$h = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{q^4}{12r^4}, \qquad f = \frac{4r^4 h}{(2r^2 - q^2)^2}.$$

$$A = \left(\frac{q}{r} - \frac{q^3}{6r^3}\right)dt, \qquad d\phi = \sqrt{\frac{-q^2}{2\gamma r^2 f}}dr.$$

- To have a real ϕ we take $\gamma < 0$.

- This solution is asymptotically flat and horizon is at $h = f = 0$,

$$r_{\mathsf{H}} - 2m + \frac{q^2}{r_{\mathsf{H}}} - \frac{q^4}{12{r_{\mathsf{H}}}^3} = 0.$$

- While the derivative of the scalar field diverges at the horizon,

$$\gamma\partial^\mu\phi\partial_\mu\phi = \frac{-q^2}{2r^2}, \qquad \mathcal{G} = 1 + 2\gamma\mathcal{X} = 1 + \frac{q^2}{2r^2}$$

  is finite at the horizon.

- Standard methods for calculating conserved charges yields

$$M = \frac{m}{G_{\mathsf{N}}}, \qquad Q = \frac{q}{G_{\mathsf{N}}}$$

- Surface gravity and horizon electric potential are

$$\kappa = \frac{2r_{\text{H}}^2 - q^2}{4r_{\text{H}}^3}, \qquad \Phi_{\text{H}} = \frac{q}{r_{\text{H}}} - \frac{q^3}{6r_{\text{H}}^3}.$$

- With the Hawking temperature $T_0 = \frac{\kappa}{2\pi}$ and the Wald entropy, which yields the usual area law for this example

$$S_{\text{BH}}{}^{\text{Wald}} = \pi r_{\text{H}}^2 / G_{\text{N}},$$

the first law

$$\delta S = \frac{1}{T_0}(\delta M - \Phi_{\text{H}} \delta Q)$$

does not hold.

- Also, with $T_{\text{BH}} = T_0$, the entropy obtained in the SPSM is not integrable over parameters m and q of the solution. This can be easily seen by replacing all terms in RHS of δS above in terms of m, q and observe that it is not variation of some $S(m, q)$.

- However the temperature,

$$T_{\mathsf{BH}} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\Big|_{\mathsf{H}} T_0 = \left(1 - \frac{q^2}{2r_{\mathsf{H}}^2}\right) T_0 = \frac{\left(1 - \frac{q^2}{2r_{\mathsf{H}}^2}\right)^2}{4\pi r_{\mathsf{H}}},$$

  makes the entropy integrable, which for this example is given by the usual Bekenstein-Hawking entropy $S_{\mathsf{BH}} = \pi r_{\mathsf{H}}^2/G_{\mathsf{N}}$.

- With this entropy and temperature, it is immediate to verify that the first law takes the usual form,

$$T_{\mathsf{BH}}\delta S_{\mathsf{BH}} = \delta M - \Phi_{\mathsf{H}}\delta Q.$$

- Note that

$$T_{\mathsf{BH}} \leq T_0, \qquad c_g^2\Big|_{\mathcal{H}} = \frac{1 - \frac{q^2}{2{r_{\mathsf{H}}}^2}}{1 + \frac{q^2}{2{r_{\mathsf{H}}}^2}} < 1.$$

Example 2: Rotating BTZ-Horndeski black holes

- Consider a 3d AdS-Horndeski theory,

$$\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}}\Big(R - 2\Lambda - 2(\alpha g^{\mu\nu} - \gamma G^{\mu\nu})\partial_\mu\phi\partial_\nu\phi\Big)$$

with arbitrary Λ and $\alpha, \gamma < 0$.

- It admits a rotating neutral BTZ-like black hole solution:

$$ds^2 = -h\ dt^2 + \frac{dr^2}{h} + r^2(d\varphi - \frac{j}{r^2}dt)^2,$$

$$h = -m + \frac{\alpha r^2}{\gamma} + \frac{j^2}{r^2}, \quad d\phi = \sqrt{\frac{-(\alpha + \gamma\Lambda)}{2\alpha\gamma h}}\ dr$$

where (m, j) are solution parameters.

- Mass, angular momentum, horizon angular velocity, surface gravity, and horizon radii for this solution are

$$M = \frac{(\alpha - \Lambda\gamma)m}{16\alpha G_{\mathrm{N}}}, \qquad J = \frac{(\alpha - \Lambda\gamma)j}{8\alpha G_{\mathrm{N}}}, \qquad \Omega_{\pm} = \frac{j}{r_{\pm}^2},$$

$$\boldsymbol{\kappa}_{\pm} = \frac{\alpha(r_{+}^2 - r_{-}^2)}{\gamma r_{\pm}}, \qquad r_{\pm}^2 = \frac{\gamma m \pm \sqrt{\gamma^2 m^2 - 4\gamma\alpha j^2}}{2\alpha}.$$

- In order to have a well-defined BH solution, $\alpha/\gamma > 0$ and $\Lambda < -\alpha/\gamma$.

- Using our formula for the temperature,

$$T_{\mathrm{BH}} = \left(\frac{\alpha - \Lambda\gamma}{2\alpha}\right) T_0, \qquad T_0 = \frac{\boldsymbol{\kappa}}{2\pi}.$$

- Using T_{BH}, SPSM yields $S_{\mathsf{BH}} = 2\pi r_{\mathsf{H}}/(4G_{\mathsf{N}})$ as the entropy.

- The above satisfy the first law for each one of the horizons

$$T_{\mathsf{BH}}\delta S_{\mathsf{BH}} = \delta M - \Omega_{\mathsf{H}}\delta J.$$

- Other examples may be found in our paper [arXiv:2005.12985].

- These examples include all known “non-trivial” Horndeski BH’s.

- For all of these examples SPSM yields an integrable entropy, and the correct first law once we use $T_{\mathsf{gravitons}}$ as the temperature.

Summary & Concluding Remarks

⊛ Wald entropy formula,

$$S^{\mathrm{W}}_{\mathrm{BH}} = \oint_{\mathcal{H}} \; \frac{1}{2}\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\lambda}} \epsilon_{\rho\lambda\alpha_1\alpha_2\cdots\alpha_{d-2}} \; \left(\mathrm{d}(\frac{2\pi\zeta_{\mathrm{H}}}{\boldsymbol{\kappa}})\right)_{\mu\nu}$$

established entropy of a black hole with a bifurcate Killing horizon as the Noether charge associated with the Killing vector field generating the horizon ζ_{H} normalized by the Bekenstein-Hawking entropy:

$$\xi_{\mathrm{H}} := \frac{\zeta_{\mathrm{H}}}{T_0}, \qquad T_0 := \frac{\boldsymbol{\kappa}}{2\pi}.$$

⊛ It is very elegant and seemed to always work prior to our work.

⊛ Wald entropy has ambiguities which had been discussed to "always" vanish for a bifurcate Killing horizon.

⊛ We discussed there are cases for which this does not necessarily hold.

⊛ For Horndeski theory we showed $\boldsymbol{W}$ ambiguity need not vanish.

⊛ To avoid the ambiguities we suggested to use solution phase space method to compute entropy variation.

⊛ In the SPSM we need to check integrability of the entropy variation.

⊛ We studied several examples of Horndeski black holes, solutions to,

$$\mathcal{L}_{\text{Horn.}} = \mathcal{G}_2 + \mathcal{G}_3 + (\mathcal{G} - \mathcal{G}'\mathcal{X})R + \mathcal{G}' G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

with $\mathcal{G}_2, \mathcal{G}$ being functions of $\mathcal{X} = -(\partial\phi)^2/2$ and $G^{\mu\nu}$ is Einstein tensor.

⊛ We observed that the SPSM entropy become integrable if we change the normalization of $\xi_{\rm H}$ to

$$\xi_{\rm H} := \frac{\zeta_{\rm H}}{T_{\rm BH}}, \qquad T_{\rm BH} := \frac{\boldsymbol{\kappa}}{2\pi}\,(\mathcal{G} - 2\mathcal{G}'\mathcal{X}),$$

computed on the solution at the horizon $\mathcal{H}$.

⊛ The issue with Wald entropy hints to redefinition of BH temperature.

⊛ BH temperature should be a geometric quantity associated with BH horizon. We hence searched for such an understanding.

⊛ Gravitons on Horndeski backgrounds move with a speed different than speed of light.

⊛ Gravitons on Horndeski backgrounds move on null geodesics of a disformal metric:

$$\mathfrak{g}_{\mu\nu} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')g_{\mu\nu} - \mathcal{G}'\partial_\mu\phi\partial_\nu\phi$$

and hence for BH backgrounds

$$c_g^2 = \begin{cases} \frac{\mathcal{G}-2\mathcal{X}\mathcal{G}'}{\mathcal{G}} & \text{for gravitons moving along } \phi_\mu \\ \frac{\mathcal{G}}{\mathcal{G}} = 1 & \text{for gravitons moving normal to } \phi_\mu \end{cases}$$

⊛ Moreover,

$$\mathrm{d}\zeta_{\mathsf{H}} = 2\boldsymbol{\kappa}_g\ \boldsymbol{\epsilon}, \qquad \boldsymbol{\kappa}_g = \boldsymbol{\kappa}\ (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\ ,$$

and hence

$$T_{\mathsf{gravitons}} := \frac{\boldsymbol{\kappa}_g}{2\pi} = (\mathcal{G} - 2\mathcal{X}\mathcal{G}')\ T_0, \qquad T_0 = \frac{\boldsymbol{\kappa}}{2\pi}$$

⊛ $T_{\mathsf{BH}} = T_{\mathsf{gravitons}}\big|_{\mathcal{H}}$ yields an integrable entropy and a well-behaved first law.

⊛ BH temperature is a geometric quantity associated with the gravitons near the horizon.

⊛ Recalling Wald's formula (or its SPSM variant) BH entropy is also depending only on the gravity sector of the Lagrangian.

⊛ For theories which do not respect Einstein's or Strong Equivalence Principle we need to revise the expressions for entropy or temperature.

⊛ Our observations and results touches upon the building principles semiclassical aspects of BH physics are stemming from.

It disentangles roles of general covariance, local Lorentz symmetry, various versions and statements of equivalence principle, background independence and ....

⊛ On the second law of BH thermodynamics

- Assigning a temperature different than $\kappa/(2\pi)$ raises the question about generalized second law of thermodynamics.

- Consider lump of gas of photons of energy $\delta E \ll m_{\mathrm{BH}}$ at temperature T_γ falling into the hole. The first law implies

$$T_{\mathrm{BH}} \Delta S_{\mathrm{BH}} = \delta E = \frac{3}{4} T_\gamma S_\gamma,$$

where ΔS_{BH} is the change in the entropy of the hole due to the fall and S_γ is the entropy of the photon lump.

- The second law requires,

$$\Delta S_{\rm BH} \geq S_\gamma \quad \text{or} \quad T_{\rm BH} \leq \frac{3}{4} T_\gamma .$$

- For a photon to be absorbed into the hole its wave-length should be smaller than $4r_{\rm H}$ and hence $T_\gamma \sim (4r_{\rm H})^{-1}$.

- Therefore, a sufficient but not necessary condition for the second law is $T_{\rm BH} \leq \frac{3}{16 r_{\rm H}}$, which is satisfied for all examples we have explored.

- There are various arguments supporting/deriving the second law.

- One should of course make sure our general picture and analysis guarantees the generalized second law beyond known Horndeski examples discussed above.

⊛ Associating the temperature and entropy only to gravitational sector, may remedy the species problem.

⊛ One should still explore and establish further our proposal by

- repeating Hawking's analysis,

- Unruh's derivation of the BH temperature,

- Gibbons-Hawking Euclidean on-shell action computation and deriving the free energy.

- Explore more closely the role of scalar field in the gravity sector, e.g. by analyzing Einstein-Æther theories where there could be spontaneous breaking of Lorentz symmetry in the near horizon geometry yielding a direction-dependent speed of gravitons.

Laws of BH thermodynamics are deeply rooted in the fabric of gravity theories.

They seem to be deeper than Equivalence Principle or general covariance.

Horndeski BH examples may be a window to further explore these deeper roots.

Thank You For Your Attention